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All null supersymmetric backgrounds of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets

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Abstract: The lightlike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to an arbitrary number of abelian vector multiplets are classified using spinorial geometry techniques. The solutions fall into two classes, depending on whether the Killing spinor is constant or not. In both cases, we give explicit examples of supersymmetric backgrounds. Among these BPS solutions, which preserve one quarter of the supersymmetry, there are gravitational waves propagating on domain walls or on bubbles of nothing that asymptote to AdS_4. Furthermore, we obtain the additional constraints obeyed by half-supersymmetric vacua. These are divided into four categories, that include bubbles of nothing which are asymptotically AdS_4, pp-waves on domain walls, $\mathrm{AdS}_3 \times \mathbb{R}$, and spacetimes conformal to AdS_3 times an interval.

Keywords: Superstring Vacua, Black Holes, Supergravity Models.

Contents

1. Introduction

Supersymmetric solutions to supergravity theories have played, and continue to play, an important role in string- and M-theory developments. This makes it desirable to obtain a complete classification of BPS solutions to various supergravities in diverse dimensions. Progress in this direction has been made in the last years using the mathematical concept of G-structures [1]. The basic strategy is to assume the existence of

at least one Killing spinor ϵ obeying $\mathcal{D}_\mu \epsilon = 0$, and to construct differential forms as bilinears from this spinor. These forms, which define a preferred G-structure, obey several algebraic and differential equations that can be used to deduce the metric and the other bosonic supergravity fields. Using this framework, a number of complete classifications [2–4] and many partial results (see e.g. [5–17] for an incomplete list) have been obtained. By complete we mean that the most general solutions for all possible fractions of supersymmetry have been obtained, while for partial classifications this is only available for some fractions. Note that the complete classifications mentioned above involve theories with eight supercharges and holonomy $H = \mathrm{SL}(2,\mathbb{H})$ of the supercurvature $R_{\mu\nu} = \mathcal{D}_{[\mu}\mathcal{D}_{\nu]}$, and allow for either half- or maximally supersymmetric solutions.

An approach which exploits the linearity of the Killing spinors has been proposed [19] under the name of spinorial geometry. Its basic ingredients are an explicit oscillator basis for the spinors in terms of forms and the use of the gauge symmetry to transform them to a preferred representative of their orbit. While the equivalent G-structure technique leads to nonlinear equations which might be difficult to interpret and to solve in some cases, the spinorial geometry approach permits to construct a linear system for the background fields from any (set of) Killing spinor(s) [20]. This method has proven fruitful in e.g. the challenging case of IIB supergravity [21–23]. In addition, it has been adjusted to impose 'near-maximal' supersymmetry and thus has been used to rule out certain large fractions of supersymmetry [24–28]. Finally, a complete classification for type I supergravity in ten dimensions has been obtained in [29], and all half-supersymmetric backgrounds of $\mathcal{N} = 2$, $D = 5$ gauged supergravity coupled to abelian vector multiplets were determined in [30, 31]. Spinorial geometry was also applied to de Sitter supergravity [32], where interesting mathematical structures like hyper-Kähler manifolds with torsion emerge.

In the present paper we shall finish the classification of supersymmetric solutions in four-dimensional $\mathcal{N} = 2$ matter-coupled U(1)-gauged supergravity initiated in [33], generalizing thus the simpler cases of $\mathcal{N} = 1$, considered recently in [34, 35], and minimal $\mathcal{N} = 2$, where a full classification is available both in the ungauged [36] and gauged theories [37]. A strong motivation for our work comes from the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, which has been attracting much attention in the last months, after the discovery of superconformal field theories describing coincident M2-branes [38, 39]. In this context, supergravity vacua with less supersymmetry correspond on the CFT side to nonzero vacuum expectation values of certain operators, or to deformations of the CFT. Disposing of a systematic classification of supergravity vacua is thus particularly useful. Of special interest in this context are domain wall solutions interpolating between vacua preserving different amounts of supersymmetry, because they can describe

a holographic RG flow.

The case where the Killing vector constructed from the Killing spinor is timelike was considered in [33], so we will now concentrate on the null class. Note that this is more than a mere extension of [33]: The timelike case typically contains black hole solutions, while the lightlike class includes gravitational waves and domain walls, whose importance in an AdS/CFT context was just explained.

The outline of this paper is as follows. In section 2, we briefly review $\mathcal{N}=2$ supergravity in four dimensions and its matter couplings, whereas in 3 the orbits of Killing spinors are discussed. In section 4 we determine the conditions coming from a single null Killing spinor and give explicit examples for supersymmetric backgrounds. Finally, in section 5, we impose a second Killing spinor and obtain the additional constraints obeyed by half-supersymmetric solutions. It is shown that half-BPS geometries are divided into four classes, that include bubbles of nothing which are asymptotically AdS_4, pp-waves on domain walls, $\mathrm{AdS}_3 \times \mathbb{R}$, and spacetimes conformal to AdS_3 times an interval. Appendices A and B contain our notation and conventions for spinors.

2. Matter-coupled $\mathcal{N}=2$, $D=4$ gauged supergravity

In this section we shall give a short summary of the main ingredients of $\mathcal{N}=2$, $D=4$ gauged supergravity coupled to vector- and hypermultiplets [40]. Throughout this paper, we will use the notations and conventions of [41], to which we refer for more details.

Apart from the vierbein e_μ^a and the chiral gravitinos ψ_μ^i, $i=1,2$, the field content includes n_H hypermultiplets and n_V vector multiplets enumerated by $I=0,\ldots,n_V$. The latter contain the graviphoton and have fundamental vectors A_μ^I, with field strengths

$$F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + g A_\nu^K A_\mu^J f_{JK}{}^I \,.$$

The fermions of the vector multiplets are denoted as $\lambda^{\alpha i}$ and the complex scalars as z^α where $\alpha = 1,\ldots,n_V$. These scalars parametrize a special Kähler manifold, i. e. , an n_V-dimensional Hodge-Kähler manifold that is the base of a symplectic bundle, with the covariantly holomorphic sections

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \,, \qquad \mathcal{D}_{\bar\alpha}\mathcal{V} = \partial_{\bar\alpha}\mathcal{V} - \frac{1}{2}(\partial_{\bar\alpha}\mathcal{K})\mathcal{V} = 0 \,, \tag{2.1}$$

where $\mathcal{K}$ is the Kähler potential and $\mathcal{D}$ denotes the Kähler-covariant derivative[1]. $\mathcal{V}$

[1]For a generic field ϕ^α that transforms under a Kähler transformation $\mathcal{K}(z,\bar z) \to \mathcal{K}(z,\bar z) + \Lambda(z) + \bar\Lambda(\bar z)$ as $\phi^\alpha \to e^{-(p\Lambda + q\bar\Lambda)/2}\phi^\alpha$, one has $\mathcal{D}_\alpha \phi^\beta = \partial_\alpha\phi^\beta + \Gamma^\beta{}_{\alpha\gamma}\phi^\gamma + \frac{p}{2}(\partial_\alpha\mathcal{K})\phi^\beta$. $\mathcal{D}_{\bar\alpha}$ is defined in the same way. X^I transforms as $X^I \to e^{-(\Lambda-\bar\Lambda)/2}X^I$ and thus has Kähler weights $(p,q)=(1,-1)$.

obeys the symplectic constraint

$$\langle \mathcal{V}\,,\bar{\mathcal{V}}\rangle = X^I\bar{F}_I - F_I\bar{X}^I = i\,. \tag{2.2}$$

To solve this condition, one defines

$$\mathcal{V} = e^{\mathcal{K}(z,\bar{z})/2}v(z)\,, \tag{2.3}$$

where $v(z)$ is a holomorphic symplectic vector,

$$v(z) = \begin{pmatrix} Z^I(z) \\ \frac{\partial}{\partial Z^I}F(Z) \end{pmatrix}\,. \tag{2.4}$$

F is a homogeneous function of degree two, called the prepotential, whose existence is assumed to obtain the last expression. This is not restrictive because it can be shown that it is always possible to go in a gauge where the prepotential exists via a local symplectic transformation [41, 42][2]. The Kähler potential is then

$$e^{-\mathcal{K}(z,\bar{z})} = -i\langle v\,,\bar{v}\rangle\,. \tag{2.5}$$

The matrix $\mathcal{N}_{IJ}$ determining the coupling between the scalars z^α and the vectors A^I_μ is defined by the relations

$$F_I = \mathcal{N}_{IJ}X^J\,, \qquad \mathcal{D}_{\bar{\alpha}}\bar{F}_I = \mathcal{N}_{IJ}\mathcal{D}_{\bar{\alpha}}\bar{X}^J\,. \tag{2.6}$$

Given

$$U_\alpha \equiv \mathcal{D}_\alpha\mathcal{V} = \partial_\alpha\mathcal{V} + \frac{1}{2}(\partial_\alpha\mathcal{K})\mathcal{V}\,, \tag{2.7}$$

the following differential constraints hold:

$$\begin{aligned} \mathcal{D}_\alpha U_\beta &= C_{\alpha\beta\gamma}g^{\gamma\bar{\delta}}\bar{U}_{\bar{\delta}}\,, \\ \mathcal{D}_{\bar{\beta}}U_\alpha &= g_{\alpha\bar{\beta}}\mathcal{V}\,, \\ \langle U_\alpha\,,\mathcal{V}\rangle &= 0\,. \end{aligned} \tag{2.8}$$

Here, $C_{\alpha\beta\gamma}$ is a completely symmetric tensor which determines also the curvature of the special Kähler manifold.

We now come to the hypermultiplets. These contain scalars q^X and spinors ζ^A, where $X = 1,\ldots,4n_H$ and $A = 1,\ldots,2n_H$. The $4n_H$ hyperscalars parametrize a

[2] This need not be true for gauged supergravity, where symplectic covariance is broken [40]. However, in our analysis we do not really use that the F_I can be obtained from a prepotential, so our conclusions go through also without assuming that $F_I = \partial F(X)/\partial X^I$ for some $F(X)$. We would like to thank Patrick Meessen for discussions on this point.

quaternionic Kähler manifold, with vielbein f_X^{iA} and inverse f_{iA}^X (i. e. the tangent space is labelled by indices (iA)). From these one can construct the three complex structures

$$\vec{J}_X{}^Y = -i f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y \,, \tag{2.9}$$

with the Pauli matrices $\vec{\sigma}_i{}^j$ (cf. appendix A). Furthermore, one defines SU(2) connections $\vec{\omega}_X$ by requiring the covariant constancy of the complex structures:

$$0 = \mathfrak{D}_X \vec{J}_Y{}^Z \equiv \partial_X \vec{J}_Y{}^Z - \Gamma^W{}_{XY} \vec{J}_W{}^Z + \Gamma^Z{}_{XW} \vec{J}_Y{}^W + 2\,\vec{\omega}_X \times \vec{J}_Y{}^Z \,, \tag{2.10}$$

where the Levi-Civita connection of the metric g_{XY} is used. The curvature of this SU(2) connection is related to the complex structure by

$$\vec{R}_{XY} \equiv 2\,\partial_{[X}\vec{\omega}_{Y]} + 2\,\vec{\omega}_X \times \vec{\omega}_Y = -\frac{1}{2}\kappa^2 \vec{J}_{XY} \,. \tag{2.11}$$

Depending on whether $\kappa = 0$ or $\kappa \neq 0$ the manifold is hyper-Kähler or quaternionic Kähler respectively. In what follows, we take $\kappa = 1$.

The bosonic action of $\mathcal{N} = 2$, $D = 4$ supergravity is

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm bos} = {}& \frac{1}{16\pi G} R + \frac{1}{4}(\mathrm{Im}\,\mathcal{N})_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{8}(\mathrm{Re}\,\mathcal{N})_{IJ}\, e^{-1}\epsilon^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^J_{\rho\sigma} \,, \\ & - g_{\alpha\bar\beta} \mathcal{D}_\mu z^\alpha \mathcal{D}^\mu \bar z^{\bar\beta} - \frac{1}{2} g_{XY} \mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y - V \,, \\ & - \frac{g}{6} C_{I,JK} e^{-1} \epsilon^{\mu\nu\rho\sigma} A^I_\mu A^J_\nu (\partial_\rho A^K_\sigma - \frac{3}{8} g f_{LM}{}^K A^L_\rho A^M_\sigma) \,, \end{aligned} \tag{2.12}$$

where $C_{I,JK}$ are real coefficients, symmetric in the last two indices, with $Z^I Z^J Z^K C_{I,JK} = 0$, and the covariant derivatives acting on the scalars read

$$\mathcal{D}_\mu z^\alpha = \partial_\mu z^\alpha + g A^I_\mu k^\alpha_I(z) \,, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + g A^I_\mu k^X_I \,. \tag{2.13}$$

Here $k^\alpha_I(z)$ and $k^X_I(q)$ are Killing vectors of the special Kähler and quaternionic Kähler manifolds respectively. The potential V in (2.12) is the sum of three distinct contributions:

$$\begin{aligned} V &= g^2(V_1 + V_2 + V_3) \,, \\ V_1 &= g_{\alpha\bar\beta} k^\alpha_I k^{\bar\beta}_J \, e^{\mathcal{K}} \bar Z^I Z^J \,, \\ V_2 &= 2\, g_{XY} k^X_I k^Y_J \, e^{\mathcal{K}} \bar Z^I Z^J \,, \\ V_3 &= 4(U^{IJ} - 3\, e^{\mathcal{K}} \bar Z^I Z^J) \vec{P}_I \cdot \vec{P}_J \,, \end{aligned} \tag{2.14}$$

with

$$U^{IJ} \equiv g^{\alpha\bar\beta} e^{\mathcal{K}} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar\beta} \bar Z^J = -\frac{1}{2}(\mathrm{Im}\,\mathcal{N})^{-1|IJ} - e^{\mathcal{K}} \bar Z^I Z^J \,, \tag{2.15}$$

and the triple moment maps $\vec{P}_I(q)$. The latter have to satisfy the equivariance condition

$$\vec{P}_I \times \vec{P}_J + \frac{1}{2}\vec{J}_{XY}k_I^X k_J^Y - f_{IJ}{}^K\vec{P}_K = 0\,, \tag{2.16}$$

which is implied by the algebra of symmetries. The metric for the vectors is given by

$$\mathcal{N}_{IJ}(z,\bar{z}) = \bar{F}_{IJ} + i\frac{N_{IN}N_{JK}Z^N Z^K}{N_{LM}Z^L Z^M}\,, \qquad N_{IJ} \equiv 2\,\mathrm{Im}\,F_{IJ}\,, \tag{2.17}$$

where $F_{IJ} = \partial_I\partial_J F$, and F denotes the prepotential.

Finally, the supersymmetry transformations of the fermions to bosons are

$$\begin{aligned}
\delta\psi_\mu^i &= D_\mu(\omega)\epsilon^i - g\Gamma_\mu S^{ij}\epsilon_j + \frac{1}{4}\Gamma^{ab}F_{ab}^{-I}\epsilon^{ij}\Gamma_\mu\epsilon_j(\mathrm{Im}\,\mathcal{N})_{IJ}Z^J e^{\mathcal{K}/2}\,, && (2.18)\\
D_\mu(\omega)\epsilon^i &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\epsilon^i + \frac{i}{2}A_\mu\epsilon^i + \partial_\mu q^X\omega_{X\,j}{}^i\epsilon^j + gA_\mu^I P_{I\,j}{}^i\epsilon^j\,, && (2.19)\\
\delta\lambda_i^\alpha &= -\frac{1}{2}e^{\mathcal{K}/2}g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{Z}^I(\mathrm{Im}\,\mathcal{N})_{IJ}F_{\mu\nu}^{-J}\Gamma^{\mu\nu}\epsilon_{ij}\epsilon^j + \Gamma^\mu\mathcal{D}_\mu z^\alpha\epsilon_i + gN_{ij}^\alpha\epsilon^j\,, \\
\delta\zeta^A &= \frac{i}{2}f_X^{Ai}\Gamma^\mu\mathcal{D}_\mu q^X\epsilon_i + g\mathcal{N}^{iA}\epsilon_{ij}\epsilon^j\,,
\end{aligned}$$

where we defined

$$\begin{aligned}
S^{ij} &\equiv -P_I^{ij}e^{\mathcal{K}/2}Z^I\,, \\
N_{ij}^\alpha &\equiv e^{\mathcal{K}/2}\left[\epsilon_{ij}k_I^\alpha\bar{Z}^I - 2P_{Iij}\mathcal{D}_{\bar{\beta}}\bar{Z}^I g^{\alpha\bar{\beta}}\right]\,, \qquad \mathcal{N}^{iA} \equiv -if_X^{iA}k_I^X e^{\mathcal{K}/2}\bar{Z}^I\,.
\end{aligned}$$

In (2.19), A_μ is the gauge field of the Kähler U(1),

$$A_\mu = -\frac{i}{2}(\partial_\alpha\mathcal{K}\partial_\mu z^\alpha - \partial_{\bar{\alpha}}\mathcal{K}\partial_\mu\bar{z}^{\bar{\alpha}}) - gA_\mu^I P_I^0\,, \tag{2.20}$$

with the moment map function

$$P_I^0 = \langle T_I\mathcal{V}\,,\bar{\mathcal{V}}\rangle\,, \tag{2.21}$$

and

$$T_I\mathcal{V} \equiv \begin{pmatrix} -f_{IJ}{}^K & 0 \\ C_{I,KJ} & f_{IK}{}^J \end{pmatrix}\begin{pmatrix} X^J \\ F_J \end{pmatrix}\,. \tag{2.22}$$

The major part of this paper will deal with the case of vector multiplets only, i. e. , $n_H = 0$. Then there are still two possible solutions of (2.16) for the moment maps $\vec{P}_I$, which are called SU(2) and U(1) Fayet-Iliopoulos (FI) terms respectively [41]. Here we are interested in the latter. In this case

$$\vec{P}_I = \vec{e}\,\xi_I\,, \tag{2.23}$$

where $\vec{e}$ is an arbitrary vector in SU(2) space and ξ_I are constants for the I corresponding to U(1) factors in the gauge group. If, moreover, we assume ${f_{IJ}}^K = 0$ (abelian gauge group), and $k_I^\alpha = 0$ (no gauging of special Kähler isometries), then only the V_3 part survives in the scalar potential (2.14), and one can also choose $C_{I,JK} = 0$. Note that this case corresponds to a gauging of a U(1) subgroup of the SU(2) R-symmetry, with gauge field $\xi_I A_\mu^I$.

3. Orbits of spinors under the gauge group

A Killing spinor[3] can be viewed as an SU(2) doublet (ϵ^1, ϵ^2), where an upper index means that a spinor has positive chirality. ϵ^i is related to the negative chirality spinor ϵ_i by charge conjugation, $\epsilon_i^C = \epsilon^i$, with

$$\epsilon_i^C = \Gamma_0 C^{-1} \epsilon_i^* \,. \tag{3.1}$$

Here C is the charge conjugation matrix defined in appendix B. As ϵ^1 has positive chirality, we can write $\epsilon^1 = c1 + de_{12}$ for some complex functions c, d. Notice that $c1 + de_{12}$ is in the same orbit as 1 under Spin(3,1), which can be seen from

$$e^{\gamma\Gamma_{13}} e^{\psi\Gamma_{12}} e^{\delta\Gamma_{13}} e^{h\Gamma_{02}}\, 1 = e^{i(\delta+\gamma)} e^h \cos\psi \, 1 + e^{i(\delta-\gamma)} e^h \sin\psi \, e_{12} \,.$$

This means that we can set $c = 1$, $d = 0$ without loss of generality. In order to determine the stability subgroup of ϵ^1, one has to solve the infinitesimal equation

$$\alpha^{cd} \Gamma_{cd} 1 = 0 \,, \tag{3.2}$$

which implies $\alpha^{02} = \alpha^{13} = 0$, $\alpha^{01} = -\alpha^{12}$, $\alpha^{03} = \alpha^{23}$. The stability subgroup of 1 is thus generated by

$$X = \Gamma_{01} - \Gamma_{12} \,, \qquad Y = \Gamma_{03} + \Gamma_{23} \,. \tag{3.3}$$

One easily verifies that $X^2 = Y^2 = XY = 0$, and thus $\exp(\mu X + \nu Y) = 1 + \mu X + \nu Y$, so that X, Y generate $\mathbb{R}^2$.

Having fixed $\epsilon^1 = 1$, also ϵ_1 is determined by $\epsilon_1 = \epsilon^{1C} = e_1$. A negative chirality spinor independent of ϵ^1 is ϵ_2, which can be written as a linear combination of odd forms, $\epsilon_2 = ae_1 + be_2$, where a and b are again complex valued functions. We can now act with the stability subgroup of ϵ^1 to bring ϵ_2 to a special form:

$$(1 + \mu X + \nu Y)(ae_1 + be_2) = be_2 + [a - 2b(\mu + i\nu)]e_1 \,.$$

[3]Our conventions for spinors and their description in terms of forms can be found in appendix B.

In the case $b = 0$ this spinor is invariant, so the representative is $\epsilon^1 = 1$, $\epsilon_2 = ae_1$ (so that $\epsilon^2 = \bar{a}1$), with isotropy group $\mathbb{R}^2$. If $b \neq 0$, one can bring the spinor to the form be_2 (which implies $\epsilon^2 = -\bar{b}e_{12}$), with isotropy group $\mathbb{I}$. The representatives[4] together with the stability subgroups are summarized in table 1. Given a Killing spinor ϵ^i, one can construct the bilinear

$$V_A = A(\epsilon^i, \Gamma_A \epsilon_i)\,, \tag{3.4}$$

with the Majorana inner product A defined in (B.4), and the sum over i is understood. For $\epsilon_2 = ae_1$, V_A is lightlike, whereas for $\epsilon_2 = be_2$ it is timelike, see table 1. The existence of a globally defined Killing spinor ϵ^i, with isotropy group $G \in$ Spin(3,1), gives rise to a G-structure. This means that we have an $\mathbb{R}^2$-structure in the null case and an identity structure in the timelike case.

In U(1) gauged supergravity, the local Spin(3,1) invariance is actually enhanced to Spin(3,1) $\times$ U(1). For U(1) Fayet-Iliopoulos terms, the moment maps satisfy (2.23), where we can choose $e^x = \delta_3^x$ without loss of generality. Then, under a gauge transformation

$$A_\mu^I \to A_\mu^I + \partial_\mu \alpha^I\,, \tag{3.5}$$

the Killing spinor ϵ^i transforms as

$$\epsilon^1 \to e^{-ig\xi_I\alpha^I}\epsilon^1\,, \qquad \epsilon^2 \to e^{ig\xi_I\alpha^I}\epsilon^2\,, \tag{3.6}$$

which can be easily seen from the supercovariant derivative (cf. eq. (2.19)). Note that ϵ^1 and ϵ^2 have opposite charges under the U(1). In order to obtain the stability subgroup, one determines the Lorentz transformations that leave the spinors ϵ^1 and ϵ^2 invariant up to arbitrary phase factors $e^{i\psi}$ and $e^{-i\psi}$ respectively, which can then be gauged away using the additional U(1) symmetry. If $\epsilon_2 = 0$, one gets in this way an isotropy group generated by X, Y and Γ_{13} obeying

$$[\Gamma_{13}, X] = -2Y\,, \qquad [\Gamma_{13}, Y] = 2X\,, \qquad [X, Y] = 0\,,$$

i. e. $G \cong \mathrm{U}(1) \ltimes \mathbb{R}^2$. For $\epsilon_2 = ae_1$ with $a \neq 0$, the stability subgroup $\mathbb{R}^2$ is not enhanced, whereas the $\mathbb{I}$ of the representative $(\epsilon^1, \epsilon_2) = (1, be_2)$ is promoted to U(1) generated by $\Gamma_{13} = i\Gamma_{\bar{\bullet}\bullet}$. The Lorentz transformation matrix a_{AB} corresponding to $\Lambda = \exp(i\psi\Gamma_{\bar{\bullet}\bullet}) \in$ U(1), with $\Lambda\Gamma_B\Lambda^{-1} = a^A{}_B\Gamma_A$, has nonvanishing components

$$a_{+-} = a_{-+} = 1\,, \qquad a_{\bullet\bar{\bullet}} = e^{2i\psi}\,, \qquad a_{\bar{\bullet}\bullet} = e^{-2i\psi}\,. \tag{3.7}$$

[4]Note the difference in form compared to the Killing spinors of the corresponding theories in five and six dimensions: in six dimensions these can be chosen constant [3] while in five dimensions they are constant up to an overall function [25]. In four dimensions such a choice is generically not possible.

Finally, notice that in U(1) gauged supergravity one can choose the function a in $\epsilon_2 = ae_1$ real and positive: Write $a = R\exp(2i\delta)$, use

$$e^{\delta\Gamma_{13}}1 = e^{i\delta}1\,, \qquad e^{\delta\Gamma_{13}}ae_1 = e^{-i\delta}ae_1 = e^{i\delta}Re_1\,,$$

and gauge away the phase factor $\exp(i\delta)$ using the electromagnetic U(1).

(ϵ^1, ϵ_2)	$G \subset$ Spin(3,1)	$G \subset$ Spin(3,1) $\times$ U(1)	$V_A E^A = A(\epsilon^i, \Gamma_A \epsilon_i) E^A$
$(1,0)$	$\mathbb{R}^2$	$\mathrm{U}(1)\ltimes\mathbb{R}^2$	$-\sqrt{2}E^-$
$(1, ae_1)$	$\mathbb{R}^2$	$\mathbb{R}^2 \quad (a \in \mathbb{R})$	$-\sqrt{2}(1+a^2)E^-$
$(1, be_2)$	$\mathbb{I}$	U(1)	$\sqrt{2}(\lvert b\rvert^2 E^+ - E^-)$

Table 1: The representatives (ϵ^1, ϵ_2) of the orbits of Weyl spinors and their stability subgroups G under the gauge groups Spin(3,1) and Spin(3,1) $\times$ U(1) in the ungauged and U(1)-gauged theories, respectively. The number of orbits is the same in both theories, the only difference lies in the stability subgroups and the fact that a is real in the gauged theory. In the last column we give the vectors constructed from the spinors.

Note that in the gauged theory the presence of G-invariant Killing spinors will in general not lead to a G-structure on the manifold but to stronger conditions. The structure group is in fact reduced to the intersection of G with Spin(3,1), and hence is equal to the stability subgroup in the ungauged theory.

The representatives, stability subgroups and vectors constructed from the Killing spinors are summarized in table 1 both for the ungauged and the U(1)-gauged cases.

4. Null representative $(\epsilon^1, \epsilon_2) = (1, ae_1)$

In this section we will analyze the conditions coming from a single null Killing spinor, and determine all supersymmetric solutions in this class. We shall first keep things general, i. e. , including hypermultiplets and a general gauging, and write down the linear system following from the Killing spinor equations. This system will then be solved for the case of U(1) Fayet-Iliopoulos terms and without hypers, while the solution in the general case will be left for a future publication. As was explained before, it is always possible to choose a real and positive, so we shall set $a = e^\chi$ in what follows.

4.1 Conditions from the Killing spinor equations

From the vanishing of the hyperini variation one obtains

$$\left(f_X^{1A} + e^\chi f_X^{2A}\right)\mathcal{D}_+ q^X = 0\ , \tag{4.1}$$

$$\left(f_X^{1A} + e^\chi f_X^{2A}\right)\mathcal{D}_\bullet q^X = ig\sqrt{2}\left(e^\chi \mathcal{N}^{1A} - \mathcal{N}^{2A}\right)\ , \tag{4.2}$$

whereas the gaugino variation yields

$$
\begin{aligned}
&-e^{\chi}e^{\mathcal{K}/2}g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}(F^{-J+-}-F^{-J\bullet\overline{\bullet}}) \\
&\qquad +\sqrt{2}\mathcal{D}_{\bullet}z^{\alpha}+g(N^{\alpha}_{11}+e^{\chi}N^{\alpha}_{12}) = 0 \ , \qquad (4.3)\\
&\sqrt{2}e^{\chi}e^{\mathcal{K}/2}g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}F^{-J-\bullet}-\mathcal{D}_{+}z^{\alpha} = 0 \ , \qquad (4.4)\\
&e^{\mathcal{K}/2}g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}(F^{-J+-}-F^{-J\bullet\overline{\bullet}}) \\
&\qquad +\sqrt{2}e^{\chi}\mathcal{D}_{\bullet}z^{\alpha}+g(N^{\alpha}_{21}+e^{\chi}N^{\alpha}_{22}) = 0 \ , \qquad (4.5)\\
&\sqrt{2}e^{\mathcal{K}/2}g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}F^{-J-\bullet}+e^{\chi}\mathcal{D}_{+}z^{\alpha} = 0 \ . \qquad (4.6)
\end{aligned}
$$

It is straightforward to show that the equations (4.3)-(4.6) imply that

$$\mathcal{D}_{+}z^{\alpha} = 0 \ , \qquad (4.7)$$

$$\mathcal{D}_{\bullet}z^{\alpha} = -g\frac{N^{\alpha}_{11}+e^{\chi}N^{\alpha}_{12}+e^{\chi}N^{\alpha}_{21}+e^{2\chi}N^{\alpha}_{22}}{\sqrt{2}(1+e^{2\chi})} \ , \qquad (4.8)$$

$$g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}F^{-J-\bullet} = 0 \ , \qquad (4.9)$$

$$e^{\mathcal{K}/2}g^{\alpha\overline{\beta}}\mathcal{D}_{\overline{\beta}}\overline{Z}^{I}(\operatorname{Im}\mathcal{N})_{IJ}(F^{-J+-}-F^{-J\bullet\overline{\bullet}}) = g\frac{e^{\chi}N^{\alpha}_{11}+e^{2\chi}N^{\alpha}_{12}-N^{\alpha}_{21}-e^{\chi}N^{\alpha}_{22}}{1+e^{2\chi}} \ . \qquad (4.10)$$

Finally, from the gravitini we get

$$
\begin{aligned}
\omega^{+-}-\omega^{\bullet\overline{\bullet}} &= 2\sqrt{2}e^{\chi}e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}F^{-I+\overline{\bullet}}E^{-} \qquad (4.11)\\
&\quad +2\sqrt{2}e^{\chi}\left[ge^{-\chi}S^{11}+gS^{12}-\frac{e^{\mathcal{K}/2}}{2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}\left(F^{-I+-}-F^{-I\bullet\overline{\bullet}}\right)\right]E^{\overline{\bullet}} \\
&\quad -2\left(\mathcal{A}_{1}{}^{1}+e^{\chi}\mathcal{A}_{2}{}^{1}\right)-iA \ , \\
\omega^{+-}-\omega^{\bullet\overline{\bullet}} &= -2\sqrt{2}e^{-\chi}e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}F^{-I+\overline{\bullet}}E^{-} \qquad (4.12)\\
&\quad +2\sqrt{2}e^{-\chi}\left[gS^{12}+ge^{\chi}S^{22}+\frac{e^{\mathcal{K}/2}}{2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}\left(F^{-I+-}-F^{-I\bullet\overline{\bullet}}\right)\right]E^{\overline{\bullet}} \\
&\quad -2\left(\mathcal{A}_{2}{}^{2}+e^{-\chi}\mathcal{A}_{1}{}^{2}\right)-iA-2d\chi \ , \\
\omega^{-\bullet} &= -\sqrt{2}e^{\chi}e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}F^{-I-\bullet}E^{\overline{\bullet}} \qquad (4.13)\\
&\quad +\sqrt{2}\left[gS^{11}+ge^{\chi}S^{12}+\frac{e^{\chi}e^{\mathcal{K}/2}}{2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}\left(F^{-I+-}-F^{-I\bullet\overline{\bullet}}\right)\right]E^{-} \ , \\
\omega^{-\bullet} &= \sqrt{2}e^{-\chi}e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}F^{-I-\bullet}E^{\overline{\bullet}} \qquad (4.14)\\
&\quad +\sqrt{2}\left[ge^{-\chi}S^{12}+gS^{22}-\frac{e^{-\chi}e^{\mathcal{K}/2}}{2}(\operatorname{Im}\mathcal{N})_{IJ}Z^{J}\left(F^{-I+-}-F^{-I\bullet\overline{\bullet}}\right)\right]E^{-} \ ,
\end{aligned}
$$

with the gauged SU(2) connection

$$\mathcal{A}_{i}{}^{j} = gA^{I}P_{Ii}{}^{j}+dq^{X}\omega_{Xi}{}^{j} \ .$$

From equations (4.13) and (4.14) one obtains

$$e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^J F^{-I-\bullet} = 0\ , \tag{4.15}$$

$$e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^J \left(F^{-I+-} - F^{-I\bullet\bar{\bullet}}\right) = -2g\frac{S^{11} + S^{12}(e^\chi - e^{-\chi}) - S^{22}}{e^\chi + e^{-\chi}}\ , \tag{4.16}$$

and

$$\omega^{-\bullet} = -\frac{\mathcal{C}_1}{\sqrt{2}}E^-\ , \tag{4.17}$$

with

$$\mathcal{C}_1 = -g\frac{e^{-\chi}S^{11} + 2S^{12} + e^\chi S^{22}}{\cosh\chi}\ .$$

As the $(n_V+1)\times(n_V+1)$ matrix $(Z^I, \mathcal{D}_{\bar\alpha}\bar{Z}^I)$ is invertible [41], eqns. (4.15), (4.16) together with (4.9), (4.10) determine uniquely the fluxes $F^{-I-\bullet}$ and $F^{-I+-} - F^{-I\bullet\bar{\bullet}}$, with the result[5]

$$\begin{aligned} F^{-I-\bullet} &= 0\,, \\ F^{-I+-} - F^{-I\bullet\bar{\bullet}} &= 4g\frac{S^{11} + S^{12}(e^\chi - e^{-\chi}) - S^{22}}{e^\chi + e^{-\chi}}e^{\mathcal{K}/2}\overline{Z}^I \\ &\quad -2g\frac{N^\alpha_{11} + e^\chi N^\alpha_{12} - e^{-\chi}N^\alpha_{21} - N^\alpha_{22}}{e^\chi + e^{-\chi}}e^{\mathcal{K}/2}\mathcal{D}_\alpha Z^I\ . \end{aligned} \tag{4.18}$$

Moreover, antiselfduality implies that

$$F^{-I+\bullet} = F^{-I-\bar{\bullet}} = F^{-I+-} + F^{-I\bullet\bar{\bullet}} = 0\,,$$

so that all fluxes except $F^{-I+\bar{\bullet}} =: \psi^I$ are fixed. Using (4.18), eqns. (4.11) and (4.12) become

$$\begin{aligned} \omega^{+-} - \omega^{\bullet\bar{\bullet}} &= 2\sqrt{2}e^\chi e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^J\psi^I E^- \\ &\quad +2\sqrt{2}ge^\chi\left[\frac{(2+e^{-2\chi})S^{11} + 2e^\chi S^{12} - S^{22}}{e^\chi + e^{-\chi}}\right]E^{\bar{\bullet}} \\ &\quad -2\left(\mathcal{A}_1{}^1 + e^\chi\mathcal{A}_2{}^1\right) - iA\ , \end{aligned} \tag{4.19}$$

$$\begin{aligned} \omega^{+-} - \omega^{\bullet\bar{\bullet}} &= -2\sqrt{2}e^{-\chi}e^{\mathcal{K}/2}(\operatorname{Im}\mathcal{N})_{IJ}Z^J\psi^I E^- \\ &\quad +2\sqrt{2}ge^{-\chi}\left[\frac{2e^{-\chi}S^{12} - S^{11} + (2+e^{2\chi})S^{22}}{e^\chi + e^{-\chi}}\right]E^{\bar{\bullet}} \\ &\quad -2\left(\mathcal{A}_2{}^2 + e^{-\chi}\mathcal{A}_1{}^2\right) - iA - 2d\chi\ , \end{aligned} \tag{4.20}$$

[5] To get this, one has to use (2.15).

from which one can determine some components of the spin connection and the gauge potential $\mathcal{A}$ as follows: First of all, (2.15) permits to decompose ψ^I in a graviphoton part ψ and matter vector part ψ^α as

$$\psi^I = \mathcal{D}_\alpha X^I \psi^\alpha + i\bar{X}^I \psi \,, \tag{4.21}$$

where

$$\psi^\alpha := -2g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}^J(\mathrm{Im}\,\mathcal{N})_{JK}\psi^K \,, \qquad i\psi := -2X^J(\mathrm{Im}\,\mathcal{N})_{JK}\psi^K \,. \tag{4.22}$$

Then, the sum of the real parts of (4.19) and (4.20) yields

$$\omega^{+-} = \sqrt{2}\sinh\chi\mathrm{Im}\psi E^- - \frac{\overline{\mathcal{C}}_2}{\sqrt{2}}E^\bullet - \frac{\mathcal{C}_2}{\sqrt{2}}E^{\bar{\bullet}} + 2\sinh\chi\mathrm{Re}\mathcal{A}_1{}^2 - d\chi \,, \tag{4.23}$$

with

$$\mathcal{C}_2 = -g\frac{e^\chi S^{11} + (e^{2\chi}+e^{-2\chi})S^{12} + e^{-\chi}S^{22}}{\cosh\chi} \,.$$

On the other hand, the difference of the real parts of (4.19) and (4.20) gives

$$\mathrm{Re}\,\mathcal{A}_1{}^2 = \frac{1}{2\cosh\chi}\left(\frac{\overline{\mathcal{C}}_3}{\sqrt{2}}E^\bullet + \frac{\mathcal{C}_3}{\sqrt{2}}E^{\bar{\bullet}} - d\chi\right) - \frac{1}{\sqrt{2}}\mathrm{Im}\psi E^- \,, \tag{4.24}$$

where we defined

$$\mathcal{C}_3 = -2g(S^{11} + 2S^{12}\sinh\chi - S^{22}) \,.$$

Plugging (4.24) into (4.23) one gets

$$\omega^{+-} = -\frac{\overline{\mathcal{C}}_1}{\sqrt{2}}E^\bullet - \frac{\mathcal{C}_1}{\sqrt{2}}E^{\bar{\bullet}} - \frac{e^\chi}{\cosh\chi}d\chi \,. \tag{4.25}$$

From the sum of the imaginary parts of (4.19) and (4.20) we have

$$\omega^{\bullet\bar{\bullet}} = i\sqrt{2}\sinh\chi\mathrm{Re}\psi E^- - \frac{\overline{\mathcal{C}}_2}{\sqrt{2}}E^\bullet + \frac{\mathcal{C}_2}{\sqrt{2}}E^{\bar{\bullet}} + iA + 2i\cosh\chi\mathrm{Im}\mathcal{A}_1{}^2 \,. \tag{4.26}$$

Finally, the difference of the imaginary parts of (4.19) and (4.20) yields

$$\mathcal{A}_1{}^1 + i\sinh\chi\mathrm{Im}\mathcal{A}_1{}^2 = \frac{1}{2\sqrt{2}}(\overline{\mathcal{C}}_3E^\bullet - \mathcal{C}_3E^{\bar{\bullet}}) - \frac{i}{\sqrt{2}}\cosh\chi\mathrm{Re}\psi E^- \,. \tag{4.27}$$

Summarizing, the components $\omega^{-\bullet}$, ω^{+-} and $\omega^{\bullet\bar{\bullet}}$ are fixed by the supersymmetry conditions, while the remaining components will be determined below by imposing vanishing torsion.

In order to obtain the spacetime geometry, we consider the spinor bilinears

$$V_{\mu}{}^{i}{}_{j} = A(\epsilon^{i}, \Gamma_{\mu}\epsilon_{j}) \ , \tag{4.28}$$

where the Majorana inner product is defined in (B.4). The nonvanishing components are

$$V_{-}{}^{1}{}_{1} = -\sqrt{2} \ , \qquad V_{-}{}^{2}{}_{2} = -e^{2\chi}\sqrt{2} \ , \qquad V_{-}{}^{1}{}_{2} = V_{-}{}^{2}{}_{1} = -e^{\chi}\sqrt{2} \ . \tag{4.29}$$

This yields for the trace part

$$V_{A}E^{A} \equiv V_{A}{}^{i}{}_{i}E^{A} = -\sqrt{2}(1+e^{2\chi})E^{-} \ . \tag{4.30}$$

Using the identities

$$\omega_{Xi}{}^{j\,*} = -\omega_{Xj}{}^{i} \ , \qquad P_{Ii}{}^{j\,*} = -P_{Ij}{}^{i} \ , \tag{4.31}$$

it is straightforward to shew that the linear system (4.11) - (4.14) implies the following constraints:

$$\begin{aligned} &\partial_{+}\chi + \frac{1}{2}\omega_{+}^{+-}(1+e^{-2\chi}) = 0 \ , \qquad \partial_{-}\chi + \frac{1}{2}\omega_{-}^{+-}(1+e^{-2\chi}) = 0 \ , \qquad \omega_{+}^{-\bullet} = 0 \ , \\ &\partial_{\bullet}\chi + \frac{1}{2}(\omega_{\bullet}^{+-} - \omega_{-}^{-\bar{\bullet}})(1+e^{-2\chi}) = 0 \ , \qquad \omega_{\bullet}^{-\bar{\bullet}} = 0 \ , \qquad \omega_{\bullet}^{-\bullet} + \omega_{\bar{\bullet}}^{-\bar{\bullet}} = 0 \ . \end{aligned} \tag{4.32}$$

These equations are easily shown to be equivalent to

$$\partial_{A}V_{B} + \partial_{B}V_{A} - \omega^{C}{}_{B|A}V_{C} - \omega^{C}{}_{A|B}V_{C} = 0 \ , \tag{4.33}$$

(where $\omega^{C}{}_{B|A} = \omega_{A}^{CD}\eta_{DB}$), which means that V is Killing. Note that $V^2 = 0$, so V is lightlike.

The next step is to impose zero torsion. The torsion two-form reads

$$\begin{aligned} T^{+} &= dE^{+} + E^{+} \wedge \left(\frac{\overline{\mathcal{C}_1}}{\sqrt{2}}E^{\bullet} + \frac{\mathcal{C}_1}{\sqrt{2}}E^{\bar{\bullet}} + \frac{e^{\chi}}{\cosh\chi}d\chi\right) + \omega^{+\bar{\bullet}} \wedge E^{\bullet} + \omega^{+\bullet} \wedge E^{\bar{\bullet}} \ , \\ T^{-} &= dE^{-} - E^{-} \wedge \left(\sqrt{2}\overline{\mathcal{C}}_1 E^{\bullet} + \sqrt{2}\mathcal{C}_1 E^{\bar{\bullet}} + \frac{e^{\chi}}{\cosh\chi}d\chi\right) \ , \\ T^{\bullet} &= dE^{\bullet} + E^{-} \wedge \left(\frac{\mathcal{C}_1}{\sqrt{2}}E^{+} + i\sqrt{2}\sinh\chi \mathrm{Re}\psi E^{\bullet} + \omega^{+\bullet}\right) \\ &\qquad -E^{\bullet} \wedge \left(\frac{\mathcal{C}_2}{\sqrt{2}}E^{\bar{\bullet}} + iA + 2i\cosh\chi \mathrm{Im}\mathcal{A}_1{}^{2}\right) \ . \end{aligned}$$

From the vanishing of T^{-} one gets $E^{-} \wedge dE^{-} = 0$, so by Fröbenius' theorem there exist two functions H and u such that locally

$$E^{-} = \frac{du}{H} \ . \tag{4.34}$$

Let us introduce a coordinate v such that

$$V = \frac{\partial}{\partial v} \ .$$

Since V is proportional to E_+ as a vector, and $\langle E_+, E^-\rangle = 0$, u is independent of v, and thus can be used as a further coordinate. Taking into account that

$$\begin{aligned}
\langle V, E^+\rangle &= -\sqrt{2}(1+e^{2\chi})\langle E_+, E^+\rangle = -\sqrt{2}(1+e^{2\chi}) \ , \\
\langle V, E^\bullet\rangle &= -\sqrt{2}(1+e^{2\chi})\langle E_+, E^\bullet\rangle = 0 \ ,
\end{aligned}$$

we obtain

$$E^+{}_v = -\sqrt{2}(1+e^{2\chi}) \ , \qquad E^\bullet{}_v = E^{\bar\bullet}{}_v = 0 \ .$$

Up to now, our discussion is completely general, i. e. , it includes hypermultiplets and a general gauging. In the remainder of this paper, we shall specialize to the case without hypers and no gauging of special Kähler isometries ($k_I^\alpha = 0$). The inclusion of hypermultiplets will be studied in a forthcoming publication. This leaves two possible solutions for the moment maps [41], namely SU(2) or U(1) Fayet-Iliopoulos (FI) terms. We shall consider here the latter, which satisfy (2.23), where $e^x = \delta_3^x$ without loss of generality[6]. One has then

$$\begin{aligned}
&P_{I1}{}^1 = -P_{I2}{}^2 = i\xi_I \ , \quad P_{I1}{}^2 = P_{I2}{}^1 = 0 \ , \\
&S^{12} = S^{21} = i\xi_I Z^I e^{\mathcal{K}/2} \ , \quad S^{11} = S^{22} = 0 \ ,
\end{aligned} \tag{4.35}$$

$$N_{11}^\alpha = N_{22}^\alpha = 0 \ , \quad N_{12}^\alpha = N_{21}^\alpha = -2i\xi_I e^{\mathcal{K}/2}\mathcal{D}_{\bar\beta}\bar Z^I g^{\alpha\bar\beta} \ ,$$

as well as

$$\mathcal{A}_1{}^2 = \mathcal{A}_2{}^1 = 0 \ , \qquad \mathcal{A}_1{}^1 = -\mathcal{A}_2{}^2 = igA^I\xi_I \tag{4.36}$$

and $\mathcal{D}_\mu z^\alpha = \partial_\mu z^\alpha$. Equ. (4.18) implies then for the fluxes

$$F^I = -2ig\tanh\chi(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_J E^\bullet\wedge E^{\bar\bullet} + \psi^I E^-\wedge E^\bullet + \bar\psi^I E^-\wedge E^{\bar\bullet} \ , \tag{4.37}$$

while (4.8) leads to the flow equation

$$\partial_\bullet z^\alpha = \frac{ig\sqrt{2}e^{\mathcal{K}/2}}{\cosh\chi} g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar Z^I\xi_I \tag{4.38}$$

for the scalars.

Notice that the special $\mathrm{U}(1)\ltimes\mathbb{R}^2$ orbit with representative $(\epsilon^1, \epsilon_2) = (1, 0)$, that can be obtained in the limit $\chi \to -\infty$, cannot occur in the FI case with nontrivial scalar

[6] $e^x = \delta_3^x$ can always be achieved by a global SU(2) rotation (which is a symmetry of the theory).

fields: Multiplying (4.20) with e^χ and letting $\chi \to -\infty$ yields (with ${\mathcal{A}_1}^2 = 0$) $gS^{12} = 0$, so that either $g = 0$ (ungauged case) or $\xi_I Z^I = 0$, which implies $\xi_I \mathcal{D}_\alpha Z^I = 0$, so that the scalars are constant (cf. section 4.3.1). In the presence of hypermultiplets and general gauging however, this orbit might occur, so there would be one more representative to consider.

In order to proceed, it is convenient to distinguish two subcases, namely $d\chi = 0$ and $d\chi \neq 0$.

4.2 Constant Killing spinor, $d\chi = 0$

If $d\chi = 0$, equation (4.24) reduces to

$$\frac{1}{2\cosh\chi}\left(\frac{\overline{\mathcal{C}}_3}{\sqrt{2}}E^{\bullet} + \frac{\mathcal{C}_3}{\sqrt{2}}E^{\bar{\bullet}}\right) - \frac{1}{\sqrt{2}}\mathrm{Im}\psi E^{-} = 0 \ ,$$

and thus $\mathrm{Im}\psi = 0$ and $\mathcal{C}_3 = 0$, which implies $\chi = 0$. Let us denote the remaining two coordinates by $w, \bar{w}$ (with $\bar{w}$ the complex conjugate of w) and define $\mathcal{G} \equiv {E^+}_u$, so that the null tetrad reads

$$\begin{aligned} E^+ &= \mathcal{G}du - 2\sqrt{2}dv + {E^+}_w dw + {E^+}_{\bar{w}} d\bar{w} \ , \\ E^- &= \frac{du}{H} \ , \\ E^{\bullet} &= {E^{\bullet}}_u du + {E^{\bullet}}_w dw + {E^{\bullet}}_{\bar{w}} d\bar{w} \ . \end{aligned}$$

To simplify $E^{\bullet}$, first perform a diffeomorphism

$$w \mapsto w'(u, w, \bar{w})$$

obeying

$$ {E^{\bullet}}_w \frac{\partial w'}{\partial \bar{w}} + {E^{\bullet}}_{\bar{w}} \frac{\partial \bar{w}'}{\partial \bar{w}} = 0 \ . \tag{4.39}$$

This eliminates ${E^{\bullet}}_{\bar{w}}$. Notice that due to the Cauchy-Kovalevskaya theorem, it is always possible to solve (4.39) locally for w'[7]. Finally, the component ${E^{\bullet}}_u$ can be removed using the residual gauge freedom, given by the stability subgroup $\mathbb{R}^2$ of the null spinor. To see this, consider an $\mathbb{R}^2$ transformation with group element

$$\Lambda = 1 + \mu X + \nu Y \ ,$$

where X and Y are given in (3.3). Defining $\alpha = \mu + i\nu$, this can also be written as

$$\Lambda = 1 + \alpha\Gamma_{+\bullet} + \bar{\alpha}\Gamma_{+\bar{\bullet}} \ . \tag{4.40}$$

[7]Because ∂_v is Killing, ${E^{\bullet}}_w$ and ${E^{\bullet}}_{\bar{w}}$ can depend on v only by a common phase factor $e^{i\lambda(u,v,w,\bar{w})}$, so that a potential v-dependence drops out of (4.39).

Given the ordering $A,B = +,-,\bullet,\bar{\bullet}$, the Lorentz transformation matrix a_{AB} corresponding to $\Lambda \in \mathbb{R}^2 \subseteq \mathrm{Spin}(3,1)$ reads

$$a_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -4|\alpha|^2 & 2\bar{\alpha} & 2\alpha \\ 0 & -2\bar{\alpha} & 0 & 1 \\ 0 & -2\alpha & 1 & 0 \end{pmatrix} . \tag{4.41}$$

The transformed vierbein ${}^\alpha E^A = a^A{}_B E^B$ is thus given by

$$\begin{aligned} {}^\alpha E^+ &= E^+ + 2\bar{\alpha}E^\bullet + 2\alpha E^{\bar{\bullet}} - 4|\alpha|^2 E^- \ , \qquad {}^\alpha E^- = E^- \ , \\ {}^\alpha E^\bullet &= E^\bullet - 2\alpha E^- \ , \qquad {}^\alpha E^{\bar{\bullet}} = E^{\bar{\bullet}} - 2\bar{\alpha}E^- \ . \end{aligned} \tag{4.42}$$

Choosing $\alpha = E^\bullet{}_u / 2E^-{}_u$ eliminates $E^\bullet{}_u$, so that we can take

$$E^\bullet = E^\bullet{}_w dw \ , \qquad E^{\bar{\bullet}} = E^{\bar{\bullet}}{}_{\bar{w}} d\bar{w}$$

without loss of generality. Then the inverse tetrad reads

$$E_+ = -\frac{1}{2\sqrt{2}}\partial_v \ , \qquad E_- = H(\partial_u + \frac{\mathcal{G}}{2\sqrt{2}}\partial_v) \ , \qquad E_\bullet = \frac{1}{E^\bullet{}_w}(\partial_w + \frac{E^+{}_w}{2\sqrt{2}}\partial_v) \ . \tag{4.43}$$

In what follows we shall set $E^\bullet{}_w \equiv \rho e^{i\zeta}$.

Equ. (4.27) reduces to

$$gA^I\xi_I = -\frac{1}{\sqrt{2}}\psi E^- \ , \tag{4.44}$$

while (4.7) and (4.8) lead to

$$\partial_v z^\alpha = 0 \tag{4.45}$$

and

$$\partial_w z^\alpha = ig\sqrt{2}\xi_I e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I g^{\alpha\bar{\beta}}\rho e^{i\zeta} \tag{4.46}$$

respectively. (4.45) implies that the scalars are independent of v and thus $A_v = 0$. The vanishing of the torsion gives the missing components $\omega^{+\bullet}$ of the spin connection (that we do not list here), plus the additional constraints

$$2i(A_u + \partial_u\zeta) = -\frac{1}{H\rho^2}(E^+{}_{w,\bar{w}} - E^+{}_{\bar{w},w}) \ , \tag{4.47}$$

$$\partial_w \ln H = 2\sqrt{2}ig\,\xi_I\bar{Z}^I e^{\mathcal{K}/2}\rho e^{i\zeta} \ , \tag{4.48}$$

$$\partial_v\rho = \partial_v\zeta = 0 \ , \tag{4.49}$$

$$i(A_{\bar{w}} + \partial_{\bar{w}}\zeta) = \sqrt{2}ig\,\xi_I Z^I e^{\mathcal{K}/2}\rho e^{-i\zeta} - \partial_{\bar{w}}\ln\rho \tag{4.50}$$

on the null tetrad.

All that remains to be done at this point is to impose the Bianchi identities and the Maxwell equations, which read respectively

$$dF^I = 0 \ , \qquad d\mathrm{Re}(\mathcal{N}_{IJ}F^{+J}) = 0 \ .$$

The fluxes can be obtained by setting $\chi = 0$ in (4.37),

$$\begin{aligned} F^I &= \psi^I E^- \wedge E^\bullet + \bar{\psi}^I E^- \wedge E^{\bar{\bullet}} \\ &= \frac{du}{H} \wedge (\psi^I \rho e^{i\zeta} dw + \bar{\psi}^I \rho e^{-i\zeta} d\bar{w}) \ , \end{aligned} \tag{4.51}$$

with the selfdual part

$$F^{+I} = \bar{\psi}^I E^- \wedge E^{\bar{\bullet}} \ . \tag{4.52}$$

One finds that the Bianchi identities imply $\partial_v \psi^I = 0$ and

$$\partial_{\bar{w}}(\psi^I \rho e^{i\zeta}/H) = \partial_w(\bar{\psi}^I \rho e^{-i\zeta}/H) \ , \tag{4.53}$$

whereas the Maxwell equations give $\partial_v \mathcal{N}_{IJ} = 0$ (which is automatically satisfied since $\partial_v z^\alpha = 0$) and

$$\partial_{\bar{w}}(\bar{\mathcal{N}}_{IJ}\psi^J \rho e^{i\zeta}/H) = \partial_w(\mathcal{N}_{IJ}\bar{\psi}^J \rho e^{-i\zeta}/H) \ . \tag{4.54}$$

Note that imposing $dF^I = 0$ is actually not sufficient; we must also ensure that $\xi_I F^I = \xi_I dA^I$, because the linear combination $\xi_I A^I$ is determined by the Killing spinor equations (cf. (4.44)). This leads to the additional condition

$$g\sqrt{2}\xi_I \psi^I \rho e^{i\zeta} = H\partial_w\left(\frac{\psi}{H}\right) \ . \tag{4.55}$$

In conclusion, the null tetrad (with the exception of $E^+{}_u = \mathcal{G}$), gauge fields and scalars are determined by the coupled system (4.45)-(4.50) and (4.53)-(4.55). Finally, the wave profile $\mathcal{G}$ is fixed by the uu component of the Einstein equations, which are given in (C.1) of [33], where in our case

$$\begin{aligned} R_{uu} = &-\frac{2\mathcal{G}_{,w\bar{w}}}{H\rho^2} - \frac{(E^+{}_{w,\bar{w}} - E^+{}_{\bar{w},w})^2}{2H^2\rho^4} + \frac{E^+{}_{w,u\bar{w}} - E^+{}_{\bar{w},uw}}{H\rho^2} \\ &-2\partial_u^2 \ln\rho - 2(\partial_u \ln\rho)(\partial_u \ln(H\rho)) \\ &+\frac{1}{H\rho^2}\left[(\mathcal{G}_{,\bar{w}} - E^+{}_{\bar{w},u})\partial_w \ln H + (\mathcal{G}_{,w} - E^+{}_{w,u})\partial_{\bar{w}} \ln H\right] \\ &+\frac{2\mathcal{G}}{H\rho^2}(\partial_w\partial_{\bar{w}} \ln H - (\partial_w \ln H)(\partial_{\bar{w}} \ln H)) \ . \end{aligned} \tag{4.56}$$

Then, as was shown in [33], all other equations of motion are automatically satisfied.

Notice that the above system is Kähler-covariant, as it must be: Under a Kähler transformation

$$\mathcal{K} \to \mathcal{K} + f(z^\alpha) + \bar{f}(\bar{z}^{\bar{\alpha}}) \ , \tag{4.57}$$

the Killing spinors transform as

$$\epsilon^i \to e^{\frac{1}{4}(\bar{f}-f)}\epsilon^i \ , \qquad \epsilon_i \to e^{-\frac{1}{4}(\bar{f}-f)}\epsilon_i \ . \tag{4.58}$$

In order for our representative $(\epsilon^1, \epsilon_2) = (1, e_1)$ to be invariant, this must be compensated by a Spin$(3,1)$ transformation $\Lambda = \exp((\bar{f} - f)\Gamma_{\bullet\bar{\bullet}}/4)$. The corresponding matrix $a_{AB} \in \mathrm{SO}(3,1)$ is given in (3.7), from which we see that $E^\bullet$ takes a phase factor $\exp(-i(\bar{f} - f)/2)$, so that ζ is shifted according to

$$\zeta \to \zeta - \frac{i}{2}(\bar{f} - f) \ . \tag{4.59}$$

Taking into account (4.57), (4.59), as well as

$$Z^I \to Z^I e^{-f} \ , \qquad \psi^I = F^{-I+\bar{\bullet}} \to a^+{}_A a^{\bar{\bullet}}{}_B F^{-IAB} = \psi^I e^{-\frac{1}{2}(\bar{f}-f)} \ ,$$

it is easy to show that the system (4.45)-(4.50) and (4.53)-(4.55) is Kähler-covariant.

In what follows, we shall obtain explicit solutions in some special cases.

4.3 Explicit solutions for $d\chi = 0$

4.3.1 Constant scalars

If we assume

$$\xi_I \mathcal{D}_\alpha Z^I = 0 \ , \tag{4.60}$$

the flow equation (4.46) implies $\partial_w z^\alpha = 0$. Actually, since the scalar potential $V = g^2 V_3$ satisfies

$$\partial_{\bar{\beta}} V = 4g^2 \xi_I \xi_J \left[g^{\gamma\bar{\delta}} e^{\mathcal{K}} \mathcal{D}_\gamma Z^I \mathcal{D}_{\bar{\beta}} \mathcal{D}_{\bar{\delta}} \bar{Z}^J - 2e^{\mathcal{K}} Z^J \mathcal{D}_{\bar{\beta}} \bar{Z}^I \right] \ , \tag{4.61}$$

(4.60) forces the scalars to be constant, i. e., they do not depend on $\bar{w}$ and u either[8]. One has then $A_\mu = 0$, so that (4.50) and the complex conjugate of (4.48) give

$$\partial_{\bar{w}}(\ln \rho + i\zeta + \frac{1}{2} \ln H) = 0 \ , \tag{4.62}$$

and thus

$$\rho e^{i\zeta} \sqrt{H} = f(u, w) \ , \tag{4.63}$$

[8]This is true if the potential has no flat directions.

with $f(u,w)$ an arbitrary function. Defining $F(u,w)$ by $f = \partial_w F$ we get

$$E^\bullet = H^{-1/2}\partial_w F dw = H^{-1/2}\left[dF - \frac{\partial F}{\partial u}du\right] \ . \tag{4.64}$$

By a diffeomorphism $w' = F(u,w)$ combined with a local Lorentz transformation (4.40) to eliminate $E^\bullet{}_u$ one can thus set (after dropping the primes) $E^\bullet = H^{-1/2}dw$ without loss of generality, so that $\rho = H^{-1/2}$, $\zeta = 0$. From (4.47) one obtains

$$E^+{}_{w,\bar{w}} - E^+{}_{\bar{w},w} = 0 \ ,$$

and hence $E^+{}_w = \partial_w m$ for some real function m that can always be set to zero by shifting v and $\mathcal{G}$. Finally, (4.48) yields

$$\sqrt{H} = \mathcal{C}w + \bar{\mathcal{C}}\bar{w} + A(u) \ , \tag{4.65}$$

where we defined the constant

$$\mathcal{C} = \sqrt{2}ig\,\xi_I\bar{Z}^I e^{\mathcal{K}/2} \ ,$$

and $A(u)$ is an arbitrary real function. The metric has the form of a Lobachevski wave on AdS,

$$ds^2 = \frac{2}{H}[\mathcal{G}du^2 - 2\sqrt{2}dvdu + dwd\bar{w}] \ . \tag{4.66}$$

Note that, by shifting w, $\mathcal{G}$ and v appropriately, one can always achieve $A(u) = 0$.

To obtain the gauge fields, observe that the Bianchi identities (4.53) imply that $\partial_w(\bar{\psi}^I H^{-3/2})$ is real,

$$\partial_w(\bar{\psi}^I H^{-3/2}) = \lambda^I(u,w,\bar{w}) \ , \qquad \lambda^I = \bar{\lambda}^I \ .$$

From the Maxwell equations (4.54) one concludes that $\mathcal{N}_{IJ}\lambda^J$ must be real as well, and thus

$$(\mathrm{Im}\,\mathcal{N})_{IJ}\lambda^J = 0 \ . \tag{4.67}$$

As $\mathrm{Im}\,\mathcal{N}$ is invertible, this yields $\lambda^I = 0$, so that

$$\psi^I = H^{3/2}\rho^I(u,w) \ , \tag{4.68}$$

for some function $\rho^I(u,w)$. Taking into account (4.60), equ. (4.21) gives

$$\psi = \frac{\sqrt{2}g}{\mathcal{C}}\xi_I\psi^I = \frac{\sqrt{2}g}{\mathcal{C}}H^{3/2}\xi_I\rho^I \ .$$

Since ψ must be real, this implies that the linear combination $\xi_I \rho^I$ can depend on u only. Then, one easily shows that (4.55) is automatically satisfied. Analogous to (4.21), we can decompose

$$\rho^I = \mathcal{D}_\alpha X^I \rho^\alpha + i\bar{X}^I \rho \ , \tag{4.69}$$

where (using $\xi_I \rho^I = i\xi_I \bar{X}^I \rho$)

$$\rho = \rho(u) \ , \qquad \rho^\alpha = \rho^\alpha(u, w) \ . \tag{4.70}$$

The fluxes are thus given by

$$F^I = \rho^I du \wedge dw + \bar{\rho}^I du \wedge d\bar{w} \ , \tag{4.71}$$

with ρ^I satisfying (4.69) and (4.70). Note that this solution with constant scalars includes the one in minimal gauged supergravity found in [7].

4.3.2 Prepotential $F = -iZ^0 Z^1$

We now consider a simple model determined by the prepotential

$$F = -iZ^0 Z^1 \ , \tag{4.72}$$

that has $n_V = 1$ (one vector multiplet), and thus just one complex scalar z. Choosing $Z^0 = 1$, $Z^1 = z$ (cf. [41]), the symplectic vector v reads

$$v = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix} \ . \tag{4.73}$$

The Kähler potential, metric and kinetic matrix for the vectors are given respectively by

$$e^{-\mathcal{K}} = 2(z + \bar{z}) \ , \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} \mathcal{K} = (z + \bar{z})^{-2} \ , \tag{4.74}$$

$$\mathcal{N} = \begin{pmatrix} -iz & 0 \\ 0 & -\frac{i}{z} \end{pmatrix} \ . \tag{4.75}$$

Note that positivity of the kinetic terms in the action requires $\mathrm{Re}z > 0$. For the scalar potential one obtains

$$V = g^2 V_3 = -\frac{4g^2}{z + \bar{z}} (\xi_0^2 + 2\xi_0\xi_1 z + 2\xi_0\xi_1 \bar{z} + \xi_1^2 z\bar{z}) \ , \tag{4.76}$$

which has an extremum at $z = \bar{z} = |\xi_0/\xi_1|$. In what follows we assume $\xi_I > 0$. The Kähler U(1) is

$$A_\mu = \frac{i}{2(z+\bar{z})}\partial_\mu(z-\bar{z}) \ . \tag{4.77}$$

In order to solve the system (4.45)-(4.50) and (4.53)-(4.55) we shall take $z = \bar{z}$ (this includes the extremum of the potential, and thus the AdS vacuum) and $\zeta = E^+{}_w = E^+{}_{\bar{w}} = \psi^I = 0$. Then, $A_\mu = 0$, and the only nontrivial equations are (4.46), (4.48) and (4.50), which become

$$\partial_w z = \sqrt{2}ig\sqrt{z}(-\xi_0 + \xi_1 z)\rho \ , \tag{4.78}$$

$$\partial_w \ln H = \sqrt{2}ig\frac{\xi_0 + \xi_1 z}{\sqrt{z}}\rho \ , \tag{4.79}$$

$$\partial_{\bar{w}} \ln \rho = ig\frac{\xi_0 + \xi_1 z}{\sqrt{2z}}\rho \ . \tag{4.80}$$

(4.79) and the complex conjugate of (4.80) can be combined to give

$$\partial_w \ln(\rho\sqrt{H}) = 0 \ , \tag{4.81}$$

and hence

$$\rho\sqrt{H} = g(u,\bar{w}) \ , \tag{4.82}$$

where $g(u,\bar{w})$ denotes an arbitrary function. Because $\rho\sqrt{H}$ is real, $g(u,\bar{w})$ can depend only on u. As was explained in section 4.3.1, one can set $g(u) = 1$ without loss of generality by a combination of a diffeomorphism $w \to w/g(u)$ and a local Lorentz transformation (4.40), so that $\rho = H^{-1/2}$. From (4.78) and (4.79) we get

$$H = \frac{(-\xi_0 + \xi_1 z)^2}{z} f(u) \ , \tag{4.83}$$

with $f(u)$ an arbitrary function that we will take equal to one in the following. Since z is real, (4.78) yields $\partial_x H = 0$, where we introduced the real coordinates x, y by $w = x + iy$. Let us further assume that also $\partial_u H = 0$, so that H (and thus, by virtue of (4.83), also z) depends only on y. Then, the flow equation (4.78) together with $\rho = H^{-1/2}$ and (4.83) implies

$$z = \frac{\xi_0}{\xi_1}e^{-2\sqrt{2}gy} \ , \tag{4.84}$$

where the integration constant was chosen such that the scalar goes to its critical value for $y \to 0$. The metric becomes

$$ds^2 = \frac{1}{2\xi_0\xi_1\sinh^2\sqrt{2}gy}\left[\mathcal{G}du^2 - 2\sqrt{2}\,dudv + dx^2 + dy^2\right] \ , \tag{4.85}$$

where $\mathcal{G}$ is determined by the uu component of the Einstein equations. (4.85) describes a gravitational wave propagating on a domain wall. For $z \to \xi_0/\xi_1$ ($y \to 0$), the geometry becomes that of a wave on AdS$_4$.

Note that perhaps some of the assumptions made above (like reality of z) can be relaxed while maintaining integrability of the equations. Another possible generalization is the inclusion of nonvanishing gauge fields. This will be done in section 5.2.

4.3.3 Ungauged case

Finally let us check if we correctly reproduce the results of [4] in the ungauged case. If $g = 0$, the flow equation (4.46) implies $\partial_w z^\alpha = 0$, and thus $z^\alpha = z^\alpha(\bar{w}, u)$. Using (2.20), this gives

$$A_{\bar{w}} = -\frac{i}{2}\partial_{\bar{w}}\mathcal{K} \ ,$$

so that (4.50) leads to

$$\partial_{\bar{w}}(\ln\rho + i\zeta + \frac{1}{2}\mathcal{K}) = 0 \ . \tag{4.86}$$

This can be integrated to give

$$\rho e^{i\zeta} = e^{-\frac{1}{2}\mathcal{K} + h(w,u)} \ , \tag{4.87}$$

with $h(w, u)$ an arbitrary function that can be set to zero without loss of generality by a combination of a diffeomorphism $w \to w'(u, w)$ and a local Lorentz transformation (4.40). From (4.48) we have $\partial_w H = \partial_{\bar{w}} H = 0$, and hence $H = H(u)$ so that we can set $H = 1$ by a redefinition of u.

In conclusion, the metric is given by

$$ds^2 = 2du\left[\mathcal{G}du - 2\sqrt{2}dv + E^+{}_w dw + E^+{}_{\bar{w}} d\bar{w}\right] + 2e^{-\mathcal{K}} dw d\bar{w} \ , \tag{4.88}$$

where

$$2iA_u = -e^{\mathcal{K}}(E^+{}_{w,\bar{w}} - E^+{}_{\bar{w},w}) \ . \tag{4.89}$$

The scalars are arbitrary functions of $\bar{w}, u$, and the gauge fields read

$$F^I = e^{-\mathcal{K}/2}\psi^I du \wedge dw + e^{-\mathcal{K}/2}\bar{\psi}^I du \wedge d\bar{w} \ , \tag{4.90}$$

with ψ^I determined by

$$\begin{aligned} \partial_{\bar{w}}(e^{-\mathcal{K}/2}\psi^I) &= \partial_w(e^{-\mathcal{K}/2}\bar{\psi}^I) \ , \\ \partial_{\bar{w}}(e^{-\mathcal{K}/2}\bar{\mathcal{N}}_{IJ}\psi^J) &= \partial_w(e^{-\mathcal{K}/2}\mathcal{N}_{IJ}\bar{\psi}^J) \ . \end{aligned} \tag{4.91}$$

(4.88), (4.89), (4.90) and (4.91) exactly coincide with the equations obtained in [4] that admit pp-waves and cosmic strings as solutions. Because (4.44) implies in addition $\psi = 0$, one actually gets only a subclass of the solutions of [4]. As $\chi = 0$ is not the only case to consider, this is not surprising.

4.4 Killing spinor with $d\chi \neq 0$

In the case $d\chi \neq 0$ we can determine explicitely the function H appearing in (4.34): From equ. (4.24) one has

$$d\chi = -\sqrt{2}\cosh\chi \mathrm{Im}\psi E^- + 2ig\sqrt{2}\sinh\chi e^{\mathcal{K}/2}\xi_I(\bar{Z}^I E^\bullet - Z^I E^{\bar{\bullet}}) \; . \tag{4.92}$$

Plugging this into $T^- = 0$ we obtain

$$d\left[\left(e^{2\chi} - 1\right) E^-\right] = 0 \; ,$$

and therefore one can introduce a function u such that

$$\left(e^{2\chi} - 1\right) E^- = du \quad \Rightarrow \quad E^- = \frac{du}{e^{2\chi} - 1} \; . \tag{4.93}$$

On the other hand, (4.27) gives

$$g\mathtt{A} = -\frac{1}{\sqrt{2}}\cosh\chi \mathrm{Re}\psi E^- + g\sqrt{2}\sinh\chi e^{\mathcal{K}/2}\xi_I(\bar{Z}^I E^\bullet + Z^I E^{\bar{\bullet}}) \; , \tag{4.94}$$

where we defined

$$\mathtt{A} = A^I \xi_I \; .$$

(4.92) and (4.94) determine the components $E^\bullet$, $E^{\bar{\bullet}}$ of the null tetrad,

$$\begin{aligned} E^\bullet &= -\frac{1}{2\sqrt{2}g\sinh\chi \bar{S}^{12}}\left(\frac{i\bar{\psi}\coth\chi}{2\sqrt{2}e^\chi}du + \frac{d\chi}{2} + ig\mathtt{A}\right) \; , \\ E^{\bar{\bullet}} &= -\frac{1}{2\sqrt{2}g\sinh\chi S^{12}}\left(-\frac{i\psi\coth\chi}{2\sqrt{2}e^\chi}du + \frac{d\chi}{2} - ig\mathtt{A}\right) \; . \end{aligned}$$

We already introduced the coordinates u, v. Using (4.92) together with $V = \partial_v = -\sqrt{2}(1 + e^{2\chi})E_+$, we get $\langle \partial_v, d\chi \rangle = 0$, and thus $\partial\chi/\partial v = 0$, so that χ is independent of v. Furthermore, (4.92) and (4.93) imply that $du \wedge d\chi \neq 0$, therefore the function χ must depend nontrivially on the two remaining coordinates. This allows to choose χ as a further coordinate. Finally, the fourth coordinate will be called Ψ. Notice that due to $\langle V, E^\bullet \rangle = 0$, $\mathtt{A}$ has no v-component, $\mathtt{A}_v = 0$.

Now we employ the $\mathbb{R}^2$ stability subgroup of the null spinor (cf. (4.42)) to set ${E^\bullet}_u = {E^{\bar{\bullet}}}_u = 0$. This amounts to the choice

$$\frac{\psi\coth\chi}{2\sqrt{2}e^\chi} + g\mathtt{A}_u = 0 \; , \tag{4.95}$$

and hence $\mathrm{Im}\psi = 0$. Using also

$$E^+ = \mathcal{G}du - \sqrt{2}(1 + e^{2\chi})dv + {E^+}_\chi d\chi + {E^+}_\Psi d\Psi \; , \tag{4.96}$$

we can proceed to impose vanishing torsion. $T^\bullet = 0$ determines the following components of the spin connection:

$$\begin{aligned}
\omega_v^{\bullet +} &= 4ge^\chi S^{12} \ , \\
\omega_\chi^{\bullet +} &= -\frac{\sqrt{2}gS^{12}}{\cosh\chi}E^+{}_\chi + 2e^\chi \sinh\chi(\partial_u + iA_u + \frac{i}{\sqrt{2}}e^{-\chi}\psi)E^\bullet{}_\chi \ , \\
\omega_\Psi^{\bullet +} &= -\frac{\sqrt{2}gS^{12}}{\cosh\chi}E^+{}_\Psi + 2e^\chi \sinh\chi(\partial_u + iA_u + \frac{i}{\sqrt{2}}e^{-\chi}\psi)E^\bullet{}_\Psi \ ,
\end{aligned} \tag{4.97}$$

whereas $T^+ = 0$ gives

$$\omega_u^{\bullet +} = \frac{8ig\sinh^2\chi|S^{12}|^2}{\mathtt{A}_\Psi}\left[E^\bullet{}_\Psi(E^+{}_{u,\chi} - E^+{}_{\chi,u}) - E^\bullet{}_\chi(E^+{}_{u,\Psi} - E^+{}_{\Psi,u})\right] - \frac{\sqrt{2}gS^{12}e^{2\chi}}{\cosh\chi}E^+{}_u \ ,$$

together with the constraint

$$\begin{aligned}
&E^+{}_{\chi,\Psi} - e^\chi\cosh\chi\partial_\chi\left(\frac{E^+{}_\Psi}{e^\chi\cosh\chi}\right) \\
&+\frac{\mathtt{A}_\Psi\psi}{2\sqrt{2}g\sinh\chi S^{12}\bar{S}^{12}} - 2e^\chi\sinh\chi\epsilon^{mn}(E^\bullet{}_m\mathcal{D}_u E_{\bullet n} + E^{\bar\bullet}{}_m\mathcal{D}_u E_{\bar\bullet n}) = 0 \ ,
\end{aligned} \tag{4.98}$$

that determines $E^+{}_\chi$ and $E^+{}_\Psi$. In (4.98) we introduced the indices $m, n, \ldots = \chi, \Psi$, and the convention $\epsilon^{\chi\Psi} = 1$. The Kähler-covariant derivatives $\mathcal{D}_u$ appearing in (4.98) are defined as

$$\mathcal{D}_u E^\bullet{}_n = (\partial_u + iA_u)E^\bullet{}_n \ , \qquad \mathcal{D}_u E^{\bar\bullet}{}_n = (\partial_u - iA_u)E^{\bar\bullet}{}_n \ .$$

(As we remarked in section 4.2, in order for the spinor representative to be invariant under a Kähler transformation, one must compensate with a Spin(3,1) transformation, which acts also on $E^\bullet, E^{\bar\bullet}$).

Finally we have to ensure that the Maxwell equations and Bianchi identities hold. From $\xi_I F^I = \xi_I dA^I$ we obtain

$$\partial_v\mathtt{A}_u = 0 \ , \qquad \partial_\chi\mathtt{A}_\Psi - \partial_\Psi\mathtt{A}_\chi = -(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_I\xi_J\frac{\mathtt{A}_\Psi}{2|S^{12}|^2\sinh 2\chi} \ , \tag{4.99}$$

$$\begin{aligned}
\xi_I\psi^I = \frac{2\sqrt{2}e^\chi\sinh^2\chi}{\mathtt{A}_\Psi}\{&\mathtt{A}_{\Psi,u} - \mathtt{A}_{u,\Psi} \\
&+2ig\left[\mathtt{A}_\chi\left(\mathtt{A}_{u,\Psi} - \mathtt{A}_{\Psi,u}\right) - \mathtt{A}_\Psi\left(\mathtt{A}_{u,\chi} - \mathtt{A}_{\chi,u}\right)\right]\}\bar{X}\cdot\xi \ .
\end{aligned} \tag{4.100}$$

Imposing the Bianchi identities, $dF^I = 0$, one gets $\partial_v\psi^I = 0$ and

$$\begin{aligned}
&\frac{1}{\sinh 2\chi}\partial_u\left[\frac{(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_J}{X\cdot\xi\,\bar{X}\cdot\xi}\mathtt{A}_\Psi\right] + \frac{1}{\sqrt{2}}\partial_\chi\left[\frac{\mathtt{A}_\Psi}{e^\chi\sinh^2\chi}\mathrm{Re}\left(\frac{\psi^I}{\bar{X}\cdot\xi}\right)\right] \\
&\qquad -\frac{1}{\sqrt{2}e^\chi\sinh^2\chi}\partial_\Psi\left[\mathtt{A}_\chi\mathrm{Re}\left(\frac{\psi^I}{\bar{X}\cdot\xi}\right) + \frac{1}{2g}\mathrm{Im}\left(\frac{\psi^I}{\bar{X}\cdot\xi}\right)\right] = 0 \ .
\end{aligned} \tag{4.101}$$

The Maxwell equations $d\mathrm{Re}\left(\mathcal{N}_{IJ}F^{+J}\right) = 0$ yield in addition

$$
\begin{aligned}
\xi_I \left[E^+{}_{\chi,\Psi} - e^\chi \cosh\chi \partial_\chi \left(\frac{E^+{}_\Psi}{e^\chi \cosh\chi} \right) \right] = \\
&- \frac{e^\chi}{4g\sinh\chi} \partial_u \left[\frac{(\mathrm{Re}\,\mathcal{N})_{IJ} \left(\mathrm{Im}\,\mathcal{N}\right)^{-1|JL} \xi_L}{X\cdot\xi\,\bar{X}\cdot\xi} \mathtt{A}_\Psi \right] \\
&- \frac{e^\chi \cosh\chi}{2\sqrt{2}g} \partial_\chi \left[\frac{\mathtt{A}_\Psi}{e^\chi \sinh^2\chi} \mathrm{Re}\left(\frac{\bar{\mathcal{N}}_{IJ}\psi^J}{\bar{X}\cdot\xi} \right) \right] \\
&+ \frac{\cosh\chi}{2\sqrt{2}g\sinh^2\chi} \partial_\Psi \left[\mathtt{A}_\chi \mathrm{Re}\left(\frac{\bar{\mathcal{N}}_{IJ}\psi^J}{\bar{X}\cdot\xi} \right) + \frac{1}{2g} \mathrm{Im}\left(\frac{\bar{\mathcal{N}}_{IJ}\psi^J}{\bar{X}\cdot\xi} \right) \right] .
\end{aligned}
\tag{4.102}
$$

Notice that (4.7) implies $\partial_v z^\alpha = 0$. Using this together with the fact that ∂_v is Killing, one easily shows that all components of the vierbein do not depend on v either.

The flow equation (4.38) becomes

$$
\bar{S}^{12} \sinh 2\chi \left[\left(g\mathtt{A}_\chi + \frac{i}{2} \right) \partial_\Psi - g\mathtt{A}_\Psi \partial_\chi \right] z^\alpha = ig\mathtt{A}_\Psi e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I \xi_I \ . \tag{4.103}
$$

In conclusion, the coupled system (4.98), (4.99), (4.100), (4.101), (4.102) and (4.103) determines the components of the null tetrad (except $E^+{}_u = \mathcal{G}$), the functions ψ^I and the scalar fields z^α. The fluxes F^I are then given by (4.37). Finally, the wave profile $\mathcal{G}$ is fixed by the uu component of the Einstein equations (cf. (C.1) of [33]), where in our case

$$
\begin{aligned}
R_{uu} = &\frac{E^+{}_u E^\bullet{}_\Psi E^{\bar\bullet}{}_\Psi}{\sigma^2 e^\chi \sinh^3\chi \cosh^2\chi} + \frac{E^+{}_u \mathrm{Re}\left[\left(E^\bullet{}_{\chi,\Psi} - E^\bullet{}_{\Psi,\chi} \right) E^{\bar\bullet}{}_\Psi \right]}{\sigma^2 e^\chi \sinh^2\chi \cosh\chi} \\
&+ \frac{2}{\sigma^2} |E^\bullet{}_{\Psi,u} E^\bullet{}_\chi - E^\bullet{}_{\chi,u} E^\bullet{}_\Psi|^2 - \frac{\Upsilon^2}{8\sigma^2 e^{2\chi} \sinh^2\chi} \\
&+ \frac{E^{\bar\bullet}{}_\Psi \Phi}{\sigma^2 e^\chi \sinh\chi \sinh 2\chi} + \frac{\bar\Phi}{\sigma^2 e^\chi \sinh\chi} \left(E^\bullet{}_{\chi,\Psi} - E^\bullet{}_{\Psi,\chi} + \frac{1 + 2e^{2\chi}}{\sinh 2\chi} E^\bullet{}_\Psi \right) \\
&- \frac{1}{e^\chi \sinh\chi} \left[\frac{1}{2} \partial_u \left(\frac{\Upsilon}{\sigma} \right) + \frac{E^\bullet{}_\Psi}{\sigma e^\chi \sinh\chi} \partial_\chi \left(\frac{e^\chi \sinh\chi \bar\Phi}{\sigma} \right) - \frac{E^\bullet{}_\chi}{\sigma} \partial_\Psi \left(\frac{\bar\Phi}{\sigma} \right) \right] \\
&+ \frac{2E^+{}_u}{e^\chi \sinh\chi} \mathrm{Re}\left[\frac{E^\bullet{}_\chi}{\sigma \sinh 2\chi} \partial_\Psi \left(\frac{E^{\bar\bullet}{}_\Psi}{\sigma} \right) - \frac{E^\bullet{}_\Psi}{\sigma} \partial_\chi \left(\frac{E^{\bar\bullet}{}_\Psi}{\sigma \sinh 2\chi} \right) \right] \ ,
\end{aligned}
\tag{4.104}
$$

and we defined

$$
\begin{aligned}
\sigma &= E^\bullet{}_\chi E^{\bar\bullet}{}_\Psi - E^{\bar\bullet}{}_\chi E^\bullet{}_\Psi \ , \\
\Phi &= \left(E^+{}_{u,\Psi} - E^+{}_{\Psi,u} \right) E^\bullet{}_\chi + \left[E^+{}_{\chi,u} - e^\chi \cosh\chi \partial_\chi \left(\frac{E^+{}_u}{e^\chi \cosh\chi} \right) \right] E^\bullet{}_\Psi \ , \\
\Upsilon &= E^+{}_{\chi,\Psi} - e^\chi \cosh\chi \partial_\chi \left(\frac{E^+{}_\Psi}{e^\chi \cosh\chi} \right) + 2e^\chi \sinh\chi \partial_u \sigma \ .
\end{aligned}
$$

Then, as was shown in [33], all other equations of motion are automatically satisfied. Note that R_{uu} in (4.104) can be rewritten in a manifestly real form, but then the expression becomes considerably longer.

In the next subsection we shall obtain an explicit solution to the above equations.

4.5 Explicit solutions for $d\chi \neq 0$

If one sets $\psi^I = \mathtt{A}_u = \mathtt{A}_\chi = {E^+}_\chi = {E^+}_\Psi = 0$ and $z^\alpha = z^\alpha(\chi)$, $\mathtt{A}_\Psi = \mathtt{A}_\Psi(\chi)$, the only nontrivial equations are (4.99) and (4.103), which reduce to

$$\sinh 2\chi \partial_\chi \ln \mathtt{A}_\Psi = -\frac{(\operatorname{Im}\mathcal{N})^{-1|IJ}\xi_I\xi_J}{2|S^{12}|^2} \ , \tag{4.105}$$

$$\sinh 2\chi \partial_\chi z^\alpha = -\frac{ie^{\mathcal{K}/2}g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar Z^I\xi_I}{\bar S^{12}} \ . \tag{4.106}$$

Note that (4.106) can also be written in the form

$$\sinh 2\chi \partial_\chi z^\alpha = g^{\alpha\bar\beta}\partial_{\bar\beta}W \ , \tag{4.107}$$

with the superpotential $W = \ln(\xi_I\bar Z^I) + \mathcal{K}$.

In what follows, we solve (4.105) and (4.106) for the simple model with prepotential $F = -iZ^0Z^1$ introduced in section 4.3.2. Assuming in addition that the single scalar $z = Z^1$ is real, (4.105) and (4.106) become

$$\sinh 2\chi \partial_\chi z = -2z\frac{\xi_0 - \xi_1 z}{\xi_0 + \xi_1 z} \ , \qquad \sinh 2\chi \partial_\chi \ln \mathtt{A}_\Psi = 2\frac{\xi_0^2 + \xi_1^2 z^2}{(\xi_0 + \xi_1 z)^2} \ ,$$

with the solution

$$z_\pm = \frac{\xi_0}{\xi_1} + c\tanh\chi \pm \sqrt{c\tanh\chi\left(\frac{2\xi_0}{\xi_1} + c\tanh\chi\right)} \ , \tag{4.108}$$

$$\mathtt{A}_\Psi^\pm = \tilde c\left(\frac{\xi_0^2}{z_\pm} - \xi_1^2 z_\pm\right) \ , \tag{4.109}$$

where $c, \tilde c$ are integration constants. Finally, for the metric and the nonvanishing components of the fluxes one obtains respectively

$$\begin{aligned} ds^2 = {} & \frac{\mathcal{G}du^2}{e^\chi \sinh\chi} - 2\sqrt{2}\coth\chi du dv \\ & + \frac{z_\pm}{\sinh^2\chi(\xi_0 + \xi_1 z_\pm)^2}\left[\frac{d\chi^2}{4g^2} + (\mathtt{A}_\Psi^\pm)^2 d\Psi^2\right] \ , \end{aligned} \tag{4.110}$$

$$F^0_{\chi\Psi} = \frac{2\xi_0\tilde{c}(\xi_0 - \xi_1 z)}{z(\xi_0 + \xi_1 z)\sinh 2\chi} \; , \qquad F^1_{\chi\Psi} = \frac{2\xi_1\tilde{c}z(\xi_0 - \xi_1 z)}{(\xi_0 + \xi_1 z)\sinh 2\chi} \; . \tag{4.111}$$

Notice that $z_\pm$ in (4.108) are related by the strong-weak coupling duality

$$z \to \frac{\xi_0^2}{\xi_1^2 z} \; , \tag{4.112}$$

that sends z_+ to z_- and vice versa. (4.112) is actually a residual $\mathbb{Z}_4$ symmetry that remains of the full symplectic duality group $\mathrm{Sp}(4,\mathbb{R})$ after the gauging: In the notation of [41], it corresponds to

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{R}) \; , \tag{4.113}$$

with $A = D = 0$,

$$C = \begin{pmatrix} \xi_0/\xi_1 & 0 \\ 0 & \xi_1/\xi_0 \end{pmatrix} \; , \tag{4.114}$$

and $B = -C^{-1}$. Since $\mathcal{S}^2 = -\mathbb{I}$, this generates $\mathbb{Z}_4$. Note also that the scalar potential (4.76) as well as the vacuum values $z = \xi_0/\xi_1$ are invariant under (4.112).

Let us now briefly discuss the properties of the spacetime (4.110). Introducing the new radial coordinate $\rho = \sqrt{\coth\chi}$, we have $\rho \to \infty$ for $\chi \to 0_+$ and $\rho \to 1$ for $\chi \to \infty$. Asymptotically for $\rho \to \infty$ one has $z_\pm \to \xi_0/\xi_1$ (the vacuum), and the metric approaches

$$ds^2 \to \rho^2 \left[\mathcal{G}du^2 - 2\sqrt{2}dudv + 2\xi_1^2\tilde{c}^2 c d\Psi^2\right] + \frac{d\rho^2}{8g^2\xi_0\xi_1\rho^2} \; , \tag{4.115}$$

which represents a Lobachevski wave on AdS_4. On the other hand, for $\rho \to 1$, $z_+ \to 2c$, $z_- \to \xi_0^2/(2c\xi_1^2)$, $\mathtt{A}^\pm_\Psi$ goes to a constant, and

$$\frac{z_\pm}{\sinh^2\chi(\xi_0 + \xi_1 z_\pm)^2}\left[\frac{d\chi^2}{4g^2} + \left(\mathtt{A}^\pm_\Psi\right)^2 d\Psi^2\right] \to \frac{z_\pm}{2g^2(\xi_0 + \xi_1 z_\pm)^2}\left[dR^2 + 4g^2R^2\left(\mathtt{A}^\pm_\Psi\right)^2 d\Psi^2\right] \; ,$$

where we defined $R = \mathrm{arcosh}\rho$. From this it is evident that in order for the metric to be regular at $\rho = 1$, one must identify[9]

$$\Psi \sim \Psi + \frac{\pi}{g|\mathtt{A}^\pm_\Psi|_{\rho=1}} \; .$$

As the spacetime ends at $\rho = 1$, (4.110) can be interpreted as a (wave on a) bubble of nothing [43, 44] that asymptotes to (a wave on) AdS_4. Notice that, in order to have a well-defined limit for the case of constant scalars ($c = 0$), one must choose $\tilde{c} \propto c^{-1/2}$.

[9]The requirement that g_{uu} behaves well at $\rho = 1$ puts some additional constraints on $\mathcal{G}$.

5. Half-supersymmetric backgrounds

Let us now investigate the additional conditions satisfied by half-supersymmetric vacua. Again, we will do this separately for $d\chi \neq 0$ and $d\chi = 0$. As the stability subgroup of the first Killing spinor was already used, the second one cannot be simplified anymore, and is thus of the general form

$$\epsilon^1 = a1 + be_{12} \ , \qquad \epsilon^2 = c1 + de_{12} \ , \qquad \epsilon_1 = \bar{a}e_1 - \bar{b}e_2 \ , \qquad \epsilon_2 = \bar{c}e_1 - \bar{d}e_2 \ ,$$

where a, b, c, d are complex-valued functions.

5.1 Case $d\chi \neq 0$

From $\delta\psi^i_+ = 0$ one obtains

$$\partial_v a = 4ige^{\chi}\left(e^{\chi}\bar{d}X^I - b\bar{X}^I\right)\xi_I \ , \tag{5.1}$$

$$\partial_v b = 0 \ , \tag{5.2}$$

$$\partial_v c = 4ige^{\chi}\left(e^{-\chi}\bar{b}X^I - d\bar{X}^I\right)\xi_I \ , \tag{5.3}$$

$$\partial_v d = 0 \ , \tag{5.4}$$

while $\delta\psi^i_- = 0$ leads to (using also (5.1)-(5.4))

$$\partial_u a = \frac{i\left(a - e^{-\chi}\bar{c}\right)\psi}{2\sqrt{2}\sinh\chi} - \frac{ig\sqrt{2}e^{\chi}\left(\bar{d}X^I - e^{\chi}b\bar{X}^I\right)\xi_I E^+{}_u}{\cosh\chi} \tag{5.5}$$
$$+2igb\sqrt{2}\sinh\chi\bar{X}\cdot\xi\left[E^+{}_{\chi,u} - E^+{}_{u,\chi} + \frac{i}{\mathtt{A}_\Psi}\left(\frac{1}{2g} - i\mathtt{A}_\chi\right)\left(E^+{}_{u,\Psi} - E^+{}_{\Psi,u}\right)\right] \ ,$$

$$\partial_u b = -\frac{ie^{-\chi}}{2\sinh\chi}\left[2be^{\chi}\sinh\chi A_u - \frac{b\psi e^{-\chi}}{\sqrt{2}} + \frac{g\sqrt{2}\left(e^{-\chi}\bar{c} - a\right)X\cdot\xi}{\cosh\chi}\right] \ , \tag{5.6}$$

$$\partial_u c = \frac{ie^{-\chi}\left(\bar{a} - e^{-\chi}c\right)\psi}{2\sqrt{2}\sinh\chi} - \frac{ig\sqrt{2}\left(e^{-\chi}\bar{b}X^I - e^{2\chi}d\bar{X}^I\right)\xi_I E^+{}_u}{\cosh\chi} \tag{5.7}$$
$$+2igd\sqrt{2}\sinh\chi\bar{X}\cdot\xi\left[E^+{}_{\chi,u} - E^+{}_{u,\chi} + \frac{i}{\mathtt{A}_\Psi}\left(\frac{1}{2g} - i\mathtt{A}_\chi\right)\left(E^+{}_{u,\Psi} - E^+{}_{\Psi,u}\right)\right] \ ,$$

$$\partial_u d = -\frac{ie^{-\chi}}{2\sinh\chi}\left[2de^{\chi}\sinh\chi A_u + \frac{d\psi e^{\chi}}{\sqrt{2}} + \frac{g\sqrt{2}\left(e^{\chi}\bar{a} - c\right)X\cdot\xi}{\cosh\chi}\right] \ . \tag{5.8}$$

The integrability conditions of the system (5.1)-(5.8) imply that

$$c = e^{\chi}\bar{a} - \frac{\tau e^{\chi}}{g\sqrt{2}}\frac{X\cdot\xi}{\bar{X}\cdot\xi}\bar{b} \ , \tag{5.9}$$

$$d = e^{-\chi}\frac{X\cdot\xi}{\bar{X}\cdot\xi}\bar{b} \ , \tag{5.10}$$

where we defined

$$\tau = \frac{\cosh\chi}{X\cdot\xi}\left[-\frac{\psi e^{-\chi}}{\sqrt{2}} - 2A_u + i\partial_u \ln\left(\frac{X\cdot\xi}{\bar{X}\cdot\xi}\right)\right] \ . \tag{5.11}$$

Plugging (5.10) into (5.1) and (5.3) one gets $\partial_v a = \partial_v c = 0$ so that a, b, c and d are functions of u, χ and Ψ only. Using (5.9) and (5.10) into (5.5)-(5.8) we see that if $b \neq 0$ we have to impose[10]

$$\begin{aligned}\partial_u\tau = {} & i\coth\chi\left[-\frac{\psi e^{-\chi}}{\sqrt{2}} - A_u + \frac{ie^{-\chi}}{2\cosh\chi}\partial_u\ln\left(\frac{X\cdot\xi}{\bar{X}\cdot\xi}\right)\right]\tau \\ & -4ig^2 e^{-\chi}\sinh 2\chi \bar{X}\cdot\xi\left[\frac{e^{\chi}E^{+}{}_{u}}{\cosh\chi} + E^{+}{}_{\chi,u} - E^{+}{}_{u,\chi}\right. \\ & \left. -\frac{i}{\mathtt{A}_\Psi}\left(\frac{\tanh\chi}{2g} + i\mathtt{A}_\chi\right)\left(E^{+}{}_{u,\Psi} - E^{+}{}_{\Psi,u}\right)\right] \ .\end{aligned} \tag{5.12}$$

Making use of (5.9) and (5.10), the remaining gravitino variations $\delta\psi^i_{\bullet} = \delta\psi^i_{\bullet} = 0$ reduce to

$$\begin{aligned}\partial_\chi a = {} & \frac{ie^{\chi}}{\sqrt{2}X\cdot\xi}\left(\frac{1}{2g} - i\mathtt{A}_\chi\right)\left[i\coth\chi A_u + \frac{e^{\chi}}{2\sinh\chi}\partial_u\ln\left(\frac{X\cdot\xi}{\bar{X}\cdot\xi}\right)\right]b \\ & + \frac{e^{\chi}}{\sqrt{2}}\partial_u\left(\frac{\mathtt{A}_\chi}{X\cdot\xi}\right)b - \frac{e^{\chi}}{2g\sqrt{2}X\cdot\xi}\left(\frac{\psi e^{-2\chi}}{\sqrt{2}\sinh\chi} + i\partial_u\ln X\cdot\xi\right)b \ ,\end{aligned} \tag{5.13}$$

$$\begin{aligned}\partial_\Psi a = {} & \frac{e^{\chi}\mathtt{A}_\Psi}{\sqrt{2}X\cdot\xi}\left[i\coth\chi A_u + \frac{e^{\chi}}{2\sinh\chi}\partial_u\ln\left(\frac{X\cdot\xi}{\bar{X}\cdot\xi}\right)\right]b \\ & + \frac{e^{\chi}}{\sqrt{2}}\partial_u\left(\frac{\mathtt{A}_\Psi}{X\cdot\xi}\right)b \ ,\end{aligned} \tag{5.14}$$

$$\partial_\chi b = -\left(\coth 2\chi + iA_\chi\right)b \ , \qquad \partial_\Psi b = -iA_\Psi b \ , \tag{5.15}$$

together with

$$2iA_m = \partial_m \ln\left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right) \ , \qquad m = \chi, \Psi \ , \tag{5.16}$$

[10] One easily shows that for $b = 0$, the second Killing spinor coincides (up to a constant prefactor) with the first one, and thus is not linearly independent. In what follows we shall therefore assume $b \neq 0$.

and

$$
\begin{aligned}
\partial_\chi \tau = & \left(2\coth 2\chi + iA_\chi + 2ig\mathtt{A}_\chi\right)\tau - \frac{2ig\mathtt{A}_\chi}{X\cdot\xi}\left[-\frac{\psi e^{-\chi}}{\sqrt{2}}\cosh\chi - \cosh\chi A_u\right. \\
& \left. + \frac{i}{2}\left(e^{-\chi}\partial_u \ln X\cdot\xi - e^{\chi}\partial_u \ln \bar{X}\cdot\xi\right) + i\sinh\chi \partial_u \ln \mathtt{A}_\Psi\right] \\
& + \frac{1}{X\cdot\xi}\left[\frac{\psi e^{-\chi}}{2\sqrt{2}}\left(2\sinh\chi - \frac{1}{\sinh\chi}\right) + \sinh\chi A_u\right. \\
& \left. + \frac{i}{2}\left(e^{-\chi}\partial_u \ln X\cdot\xi + e^{\chi}\partial_u \ln \bar{X}\cdot\xi\right) + 2g\sinh\chi\left(\mathtt{A}_{\chi,u} - \mathtt{A}_\chi \partial_u \ln \mathtt{A}_\Psi\right)\right] \ ,
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
\partial_\Psi \tau = & \ i\left(A_\Psi + 2g\mathtt{A}_\Psi\right)\tau - \frac{2ig\mathtt{A}_\Psi}{X\cdot\xi}\left[-\frac{\psi e^{-\chi}}{\sqrt{2}}\cosh\chi - \cosh\chi A_u\right. \\
& \left. + \frac{i}{2}\left(e^{-\chi}\partial_u \ln X\cdot\xi - e^{\chi}\partial_u \ln \bar{X}\cdot\xi\right) + i\sinh\chi \partial_u \ln \mathtt{A}_\Psi\right] \ .
\end{aligned}
\tag{5.18}
$$

The vanishing of the gaugino variations yields the additional conditions

$$
g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}^I\left[\frac{i\xi_I\tau}{\cosh\chi} - e^{-\chi}\sqrt{2}\left(\mathrm{Im}\,\mathcal{N}\right)_{IJ}\psi^J\right] = 0 \ , \tag{5.19}
$$

$$
\partial_\bullet z^\alpha = -\frac{ig\sqrt{2}e^{\mathcal{K}/2}g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{Z}^I\xi_I}{\cosh\chi}\frac{X\cdot\xi}{\bar{X}\cdot\xi} \ , \tag{5.20}
$$

as well as $\partial_u z^\alpha = 0$. This implies $A_u = 0$, so that (5.11) simplifies to

$$
\tau = -\frac{\psi e^{-\chi}\cosh\chi}{\sqrt{2}X\cdot\xi} = 2g\mathtt{A}_u e^{\chi}\tanh\chi \ ,
$$

where in the last step we used the gauge condition (4.95). Thus, equ. (5.19) reduces to

$$
g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}^I\left[\frac{\psi\xi_I}{X\cdot\xi} - 2i\left(\mathrm{Im}\,\mathcal{N}\right)_{IJ}\psi^J\right] = 0 \ . \tag{5.21}
$$

Because $\psi = 2i\left(\mathrm{Im}\mathcal{N}\right)_{IJ}X^I\psi^J$, we have moreover

$$
X^I\left[\frac{\psi\xi_I}{X\cdot\xi} - 2i\left(\mathrm{Im}\,\mathcal{N}\right)_{IJ}\psi^J\right] = 0 \ . \tag{5.22}
$$

Since the $(n_V+1)\times(n_V+1)$ matrix $(X^I, \mathcal{D}_{\bar{\alpha}}\bar{X}^I)$ is invertible, (5.21) and (5.22) imply

$$
\psi^I = -\frac{i\psi\left(\mathrm{Im}\,\mathcal{N}\right)^{-1|IL}\xi_L}{2X\cdot\xi} \ . \tag{5.23}
$$

(5.20), together with (4.38), leads to

$$\partial_\chi z^\alpha = \frac{g^{\alpha\bar\beta}\mathcal{D}_{\bar\beta}\bar Z^I\xi_I}{\sinh 2\chi \bar Z^J\xi_J}\ , \qquad \partial_\Psi z^\alpha = 0\ , \tag{5.24}$$

so that the scalars are functions of χ only, and hence A_Ψ vanishes as well. Notice that the relations (5.16) are identically satisfied if (5.24) hold. The complex equation (5.12) boils down to

$$\partial_u \mathtt{A}_u = -2e^{-\chi}\sinh\chi X\!\cdot\!\xi\,\bar X\!\cdot\!\xi \frac{E^+{}_{u,\Psi} - E^+{}_{\Psi,u}}{\mathtt{A}_\Psi}\ , \tag{5.25}$$

$$\begin{aligned}\mathtt{A}_u^2 = 2e^{-\chi}\cosh\chi X\!\cdot\!\xi\,\bar X\!\cdot\!\xi \Bigg[&\frac{e^\chi E^+{}_u}{\cosh\chi} + E^+{}_{\chi,u} - E^+{}_{u,\chi} \\ &+\frac{\mathtt{A}_\chi}{\mathtt{A}_\Psi}\left(E^+{}_{u,\Psi} - E^+{}_{\Psi,u}\right)\Bigg]\ ,\end{aligned} \tag{5.26}$$

while (5.17) and (5.18) yield

$$\partial_\chi \mathtt{A}_u - \partial_u \mathtt{A}_\chi = -(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_I\xi_J \frac{\mathtt{A}_u}{2\sinh 2\chi X\!\cdot\!\xi\,\bar X\!\cdot\!\xi}\ , \tag{5.27}$$

$$\partial_\Psi \mathtt{A}_u - \partial_u \mathtt{A}_\Psi = 0\ . \tag{5.28}$$

Using (5.23) and (5.28), it is easy to show that (5.27) is equivalent to (4.100) that follows from $\xi_I F^I = \xi_I dA^I$. Moreover, the Bianchi identities (4.101) are automatically satisfied once (5.23) and (5.28) hold. Note also the similarity between (5.27) and (4.99). Equ. (4.98) becomes

$$\begin{aligned}&E^+{}_{\chi,\Psi} - e^\chi\cosh\chi\partial_\chi\left(\frac{E^+{}_\Psi}{e^\chi\cosh\chi}\right) \\ &-\frac{\mathtt{A}_\Psi\mathtt{A}_u e^\chi}{\cosh\chi X\!\cdot\!\xi\,\bar X\!\cdot\!\xi} - \frac{e^\chi}{2\sinh\chi X\!\cdot\!\xi\,\bar X\!\cdot\!\xi}\left[\mathtt{A}_\chi\mathtt{A}_{\Psi,u} - \mathtt{A}_\Psi\mathtt{A}_{\chi,u}\right] = 0\ .\end{aligned} \tag{5.29}$$

Making use of this, together with (4.99), (5.27) and (5.28), one shows that the Maxwell equations (4.102) are identically satisfied.

(4.99), (5.27) and (5.28) can be easily integrated, with the result

$$\mathtt{A}_\mu = (X\!\cdot\!\xi\,\bar X\!\cdot\!\xi\tanh\chi)^{1/2}\partial_\mu\Xi(u,\chi,\Psi)\ , \tag{5.30}$$

where $\Xi(u,\chi,\Psi)$ denotes an arbitrary function obeying $\partial_\Psi\Xi \neq 0$[11]. Furthermore, (5.25), (5.26) and (5.29) imply for E^+

$$\begin{aligned}\frac{E^+{}_u}{e^\chi\cosh\chi} - \frac{\mathtt{A}_u^2}{2X\!\cdot\!\xi\,\bar X\!\cdot\!\xi\sinh 2\chi} &= \partial_u\Lambda\ , \\ \frac{E^+{}_m}{e^\chi\cosh\chi} - \frac{\mathtt{A}_u\mathtt{A}_m}{X\!\cdot\!\xi\,\bar X\!\cdot\!\xi\sinh 2\chi} &= \partial_m\Lambda\ , \qquad m = \chi,\Psi\ ,\end{aligned} \tag{5.31}$$

[11] $\mathtt{A}_\Psi = 0$ would lead to a singular metric.

with $\Lambda(u, \chi, \Psi)$ again a function that can be chosen at will. By shifting v one can set $\Lambda = 0$ without loss of generality. Finally, we may employ the residual gauge freedom related to the choice of the coordinate Ψ that consists in sending $\Psi \mapsto f(u, \chi, \Psi)$, where the only constraint on f is $\partial_\Psi f \neq 0$[12]. Choosing $f = \Xi$ we have then

$$\mathtt{A}_u = \mathtt{A}_\chi = 0 \ , \qquad \mathtt{A}_\Psi = (X{\cdot}\xi\, \bar{X}{\cdot}\xi \tanh\chi)^{1/2} \ , \tag{5.32}$$

and thus ${E^+}_u = {E^+}_\chi = {E^+}_\Psi = 0$. This means that the most general half-supersymmetric background in this class is given by

$$ds^2 = -2\sqrt{2}\coth\chi du dv + \frac{d\chi^2}{16g^2\sinh^2\chi X{\cdot}\xi\, \bar{X}{\cdot}\xi} + \frac{d\Psi^2}{2\sinh 2\chi} \ , \tag{5.33}$$

$$F^I = \frac{(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_J}{4\cosh^2\chi(X{\cdot}\xi\, \bar{X}{\cdot}\xi \tanh\chi)^{1/2}}\, d\Psi \wedge d\chi \ , \tag{5.34}$$

while the scalars $z^\alpha(\chi)$ follow from the flow equation (5.24). Hence, all the solutions of section 4.5 with $\mathcal{G} = 0$ are actually half-BPS.

Integration of the Killing spinor equations (5.5), (5.6), (5.13), (5.14) and (5.15) yields

$$b = b_0 \left(\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi \sinh 2\chi}\right)^{1/2} \ , \qquad a = a_0 \ , \tag{5.35}$$

where a_0, b_0 are constants. In what follows, we shall take $b_0 = 1$ without loss of generality (for $b_0 = 0$ one gets the first Killing spinor). Then the functions c and d read

$$c = e^\chi \bar{a}_0 \ , \qquad d = e^{-\chi} b \ . \tag{5.36}$$

The bilinear $V_\mu = A(\epsilon^i, \Gamma_\mu \epsilon_i)$ associated to the second covariantly constant spinor has norm squared

$$V^2 = -8\frac{(\mathrm{Re}a_0 \sinh\chi)^2 + (\mathrm{Im}a_0\cosh\chi)^2}{|\sinh\chi|\cosh\chi} \ , \tag{5.37}$$

which is in general negative, so that the solution belongs also to the timelike class studied in [33]. For $a_0 = 0$ and $\chi > 0$, the second Killing vector is given by $V = \partial_u$.

Note that the uu component of the Einstein equations is identically satisfied for the half-supersymmetric backgrounds. This is not surprising, since they belong also to the timelike class, where the Killing spinor equations imply all the equations of motion [33].

[12] As this will in general change ${E^\bullet}_u$, one must compensate by a local Lorentz transformation (4.42) in order to preserve the gauge condition ${E^\bullet}_u = 0$.

5.2 Case $d\chi = 0$

From the vanishing of the gravitino variation we obtain the system

$$
\begin{aligned}
\partial_u a &= \frac{i\psi}{\sqrt{2}H}(a-\bar{c}) - \frac{E^{+}{}_{u,w} - E^{+}{}_{w,u}}{E^{\bullet}{}_{w}} b + ig\sqrt{2}E^{+}{}_{u}(b\bar{X}^I - \bar{d}X^I)\xi_I \ , \\
\partial_u b &= -iA_u b + \frac{i\psi}{\sqrt{2}H} b + \frac{ig\sqrt{2}}{H} X{\cdot}\xi(a-\bar{c}) \ , \\
\partial_u c &= \frac{i\psi}{\sqrt{2}H}(\bar{a}-c) - \frac{E^{+}{}_{u,w} - E^{+}{}_{w,u}}{E^{\bullet}{}_{w}} d - ig\sqrt{2}E^{+}{}_{u}(\bar{b}X^I - d\bar{X}^I)\xi_I \ , \\
\partial_u d &= -iA_u d - \frac{i\psi}{\sqrt{2}H} d - \frac{ig\sqrt{2}}{H} X{\cdot}\xi(\bar{a}-c) \ ,
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
\partial_v a &= -4ig(b\bar{X}^I - \bar{d}X^I)\xi_I \ , \\
\partial_v b &= \partial_v d = 0 \ , \\
\partial_v c &= 4ig(\bar{b}X^I - d\bar{X}^I)\xi_I \ ,
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
\partial_w a &= -\frac{i\psi}{\sqrt{2}} E^{\bullet}{}_{w}\bar{d} + ig\sqrt{2}E^{+}{}_{w}(b\bar{X}^I - \bar{d}X^I)\xi_I \ , \\
\partial_w b &= -iA_w b - ig\sqrt{2}X{\cdot}\xi E^{\bullet}{}_{w}\bar{d} \ , \\
\partial_w c &= \frac{i\psi}{\sqrt{2}} E^{\bullet}{}_{w}\bar{b} - ig\sqrt{2}E^{+}{}_{w}(\bar{b}X^I - d\bar{X}^I)\xi_I \ , \\
\partial_w d &= -iA_w d - ig\sqrt{2}X{\cdot}\xi E^{\bullet}{}_{w}\bar{b} \ ,
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\partial_{\bar{w}} a &= -\left(iA_u - \partial_u \ln E^{\bar{\bullet}}{}_{\bar{w}}\right) E^{\bar{\bullet}}{}_{\bar{w}} H b - ig\sqrt{2}X{\cdot}\xi E^{\bar{\bullet}}{}_{\bar{w}}(a-\bar{c}) + ig\sqrt{2}E^{+}{}_{\bar{w}}(b\bar{X}^I - \bar{d}X^I)\xi_I \ , \\
\partial_{\bar{w}} b &= -\left(iA_{\bar{w}} - ig\sqrt{2}X{\cdot}\xi E^{\bar{\bullet}}{}_{\bar{w}}\right) b \ , \\
\partial_{\bar{w}} c &= -\left(iA_u - \partial_u \ln E^{\bar{\bullet}}{}_{\bar{w}}\right) E^{\bar{\bullet}}{}_{\bar{w}} H d + ig\sqrt{2}X{\cdot}\xi E^{\bar{\bullet}}{}_{\bar{w}}(\bar{a}-c) - ig\sqrt{2}E^{+}{}_{\bar{w}}(\bar{b}X^I - d\bar{X}^I)\xi_I \ , \\
\partial_{\bar{w}} d &= -\left(iA_{\bar{w}} - ig\sqrt{2}X{\cdot}\xi E^{\bar{\bullet}}{}_{\bar{w}}\right) d \ ,
\end{aligned}
\tag{5.41}
$$

while the gaugino supersymmetry transformations yield

$$
\partial_u z^\alpha = \frac{ig\sqrt{2}}{2H} e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I \xi_I \left(\frac{\bar{a}-c}{\bar{b}} - \frac{a-\bar{c}}{\bar{d}}\right) \ , \tag{5.42}
$$

$$
\partial_{\bar{w}} z^\alpha = -ig\sqrt{2} e^{\mathcal{K}/2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I \xi_I \frac{b}{\bar{d}} E^{\bar{\bullet}}{}_{\bar{w}} \ , \tag{5.43}
$$

$$
0 = g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{Z}^I \left[ig\xi_I \left(\frac{\bar{a}-c}{\bar{b}} + \frac{a-\bar{c}}{\bar{d}}\right) - 2(\mathrm{Im}\,\mathcal{N})_{IJ} \psi^J \frac{b}{\bar{d}} \right] \ , \tag{5.44}
$$

as well as $\bar{b}b = \bar{d}d$. Note that we assume that both b and d are nonvanishing, because for $b = 0$ or $d = 0$ the only solution to (5.38)-(5.41) is the first Killing spinor. From (4.46), (5.42) and (5.43) one gets

$$\begin{aligned}
\partial_u(X{\cdot}\xi\,\bar{X}{\cdot}\xi) &= \frac{ig}{\sqrt{2}H} g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}}\bar{X}^J \xi_I\xi_J \left(\bar{X}{\cdot}\xi + X{\cdot}\xi\frac{\bar{d}}{b}\right)\left(\frac{\bar{a}-c}{\bar{b}} - \frac{a-\bar{c}}{\bar{d}}\right) \;, \\
\partial_w(X{\cdot}\xi\,\bar{X}{\cdot}\xi) &= ig\sqrt{2}E^{\bullet}{}_w g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}}\bar{X}^J \xi_I\xi_J \left(\bar{X}{\cdot}\xi + X{\cdot}\xi\frac{\bar{d}}{b}\right) \;,
\end{aligned} \tag{5.45}$$

that will be useful later. Equ. (5.44) allows to determine ψ^I,

$$\psi^I = i\psi\bar{X}^I - ig\left(\frac{a-\bar{c}}{b} + \frac{\bar{a}-c}{d}\right) g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}}\bar{X}^J\xi_J \;. \tag{5.46}$$

The u-v, w-v and $\bar{w}$-v integrability conditions imply

$$b(\partial_\mu - iA_\mu)\bar{X}{\cdot}\xi - \bar{d}(\partial_\mu + iA_\mu)X{\cdot}\xi = 0 \;, \tag{5.47}$$

$$\psi\left(\bar{X}^I b - X^I\bar{d}\right)\xi_I = 0 \;. \tag{5.48}$$

From (5.48) we have $\psi = 0$ or

$$d = \frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}\bar{b} \;. \tag{5.49}$$

Let us first consider the latter case (5.49). Then, (5.47) gives

$$A_\mu = \frac{i}{2}\partial_\mu \ln\left(\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}\right) \;. \tag{5.50}$$

Notice that this follows also from (4.46), (5.42), (5.43) and (5.49).

Using (5.50), equ. (4.50) can be readily integrated, with the result

$$E^{\bullet}{}_w = H^{-1/2}\left(\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}\right)^{1/2} f(u,w) \;, \tag{5.51}$$

where $f(u,w)$ denotes an arbitrary function that can be set to one without loss of generality by a reasoning analogous to that following (4.63). Plugging (5.50) into (4.47) leads to $E^+{}_w = \partial_w m$, with m some real function. By shifting v and $\mathcal{G}$ appropriately, one can always achieve $m = 0$. (4.48) simplifies to

$$\partial_w\sqrt{H} = \sqrt{2}ig(X{\cdot}\xi\,\bar{X}{\cdot}\xi)^{1/2} \;, \tag{5.52}$$

which implies

$$(\partial_w + \partial_{\bar{w}})H = 0 \;, \tag{5.53}$$

so that H depends on $w-\bar{w}$ and u only. Moreover, combining (5.43) with the flow equation (4.46), we obtain

$$(\partial_w+\partial_{\bar{w}})z^\alpha = 0\ , \tag{5.54}$$

and thus the scalars are independent of $w+\bar{w}$ as well. The remaining integrability conditions for the system (5.38)-(5.41) turn out to be

$$\begin{aligned}
0 &= \left[\left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right)^{1/2}\sqrt{H}\partial_u(X\cdot\xi\,\bar{X}\cdot\xi)+\left(\psi^I-i\psi\bar{X}^I\right)\xi_I\left(\frac{2X\cdot\xi\,\bar{X}\cdot\xi}{H}\right)^{1/2}\right]b \\
&\quad+\partial_w(X\cdot\xi\,\bar{X}\cdot\xi)(a-\bar{c})\ , &(5.55)\\
0 &= -\frac{ig}{H^2}\frac{X\cdot\xi}{\bar{X}\cdot\xi}\left[\psi^I+2i\psi\bar{X}^I\right]\xi_I(a-\bar{c})+\left[\partial_w^2\mathcal{G}-\frac{i\partial_u\psi}{\sqrt{2}H}\right]b\ , &(5.56)\\
0 &= \frac{ig}{H^{3/2}}\left[\sqrt{2}H\partial_u(X\cdot\xi\,\bar{X}\cdot\xi)^{1/2}-\left(\psi^I-i\psi\bar{X}^I\right)\xi_I\left(\frac{X\cdot\xi}{\bar{X}\cdot\xi}\right)^{1/2}\right](a-\bar{c}) &(5.57)\\
&\quad-\left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right)^{1/2}\left[\frac{\psi^2}{2H^{3/2}}+\sqrt{H}\partial_w\partial_{\bar{w}}\mathcal{G}-\partial_u^2\sqrt{H}+ig(2X\cdot\xi\,\bar{X}\cdot\xi)^{1/2}(\partial_w-\partial_{\bar{w}})\mathcal{G}\right]b\ ,
\end{aligned}$$

together with

$$\left(\bar{\psi}^I\bar{X}^J+\psi^IX^J\right)\xi_I\xi_J = 0\ , \tag{5.58}$$

and

$$\left(\partial_w^2-\partial_{\bar{w}}^2\right)\mathcal{G} = 0\ . \tag{5.59}$$

Making use of (5.58) in the complex conjugate of (4.55) (recall that ψ is real) yields

$$(\partial_w+\partial_{\bar{w}})\frac{\psi}{H} = 0\ , \tag{5.60}$$

hence $\psi=\psi(w-\bar{w},u)$. Plugging the eqns. (5.45) as well as the contraction of (5.46) with ξ_I into (5.55), one finds that the latter is identically satisfied. From (5.46) and (5.56) one obtains

$$\frac{a-\bar{c}}{b}\partial_w\psi = \left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right)^{1/2}\left[\frac{iH^2\partial_w\psi\,\partial_w(\psi H^{-3/2})}{2\sqrt{2}g^2g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I\mathcal{D}_{\bar{\beta}}\bar{X}^J\xi_I\xi_J}-\sqrt{H}\partial_u\psi\right]\ , \tag{5.61}$$

and the constraint

$$\partial_w^2\mathcal{G} = -\frac{H^{1/2}\partial_w\psi\,\partial_w(\psi H^{-3/2})}{4g^2g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I\mathcal{D}_{\bar{\beta}}\bar{X}^J\xi_I\xi_J}\ , \tag{5.62}$$

where we used

$$\begin{aligned}
\left(\psi^I-i\psi\bar{X}^I\right)\xi_I &= \left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right)^{1/2}\frac{H^2}{g\sqrt{2}}\partial_w(\psi H^{-3/2})\ ,\\
\left(\psi^I+2i\psi\bar{X}^I\right)\xi_I &= \left(\frac{H\bar{X}\cdot\xi}{2X\cdot\xi}\right)^{1/2}\frac{\partial_w\psi}{g}\ ,
\end{aligned}$$

that follow from (4.55) and (5.52). Note that above we assumed that $g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}}\bar{X}^J\xi_I\xi_J$ is nonvanishing. If this expression were zero, then $\xi_I\mathcal{D}_\alpha X^I = 0$, since the Kähler metric $g_{\alpha\bar{\beta}}$ is non-degenerate. As a consequence, the scalars are constant, as can be seen from (4.61). This case was considered in section 4.3.1 and will not be pursued further here.

In addition to (5.59)and (5.62), the function $\mathcal{G}$ must obey the uu component of the Einstein equations, that becomes

$$\begin{aligned} 0 = \partial_w\partial_{\bar{w}}\mathcal{G} - \frac{\partial_u^2\sqrt{H}}{\sqrt{H}} + ig\sqrt{\frac{2X\cdot\xi\,\bar{X}\cdot\xi}{H}}(\partial_w - \partial_{\bar{w}})\mathcal{G} \\ + \frac{ig}{\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi H}}\frac{\left[\partial_u(X\cdot\xi\,\bar{X}\cdot\xi)\right]^2}{\partial_w(X\cdot\xi\,\bar{X}\cdot\xi)} - \frac{(\mathrm{Im}\,\mathcal{N})_{IJ}\bar{\psi}^I\psi^J}{H^2}\ . \end{aligned} \tag{5.63}$$

Using

$$\psi^I = \left(\frac{\bar{X}\cdot\xi}{X\cdot\xi}\right)^{1/2}\frac{H^2\partial_w(\psi H^{-3/2})g^{\gamma\bar{\delta}}\mathcal{D}_\gamma X^I\mathcal{D}_{\bar{\delta}}\bar{X}^J\xi_J}{g\sqrt{2}g^{\alpha\bar{\beta}}\mathcal{D}_\alpha X^K\mathcal{D}_{\bar{\beta}}\bar{X}^L\xi_K\xi_L} + i\psi\bar{X}^I\ , \tag{5.64}$$

following from (5.46) and (5.61), one finds that (5.63) is equivalent to the last integrability condition (5.57).

Finally we come to the Maxwell equations and Bianchi identities. As is clear from (5.64), the expression $\psi^I\rho e^{i\zeta}/H$ entering (4.53) and (4.54) depends on $w - \bar{w}$ and u only, such that $(\partial_w + \partial_{\bar{w}})(\psi^I\rho e^{i\zeta}/H) = 0$. Plugging this into (4.53), one gets

$$\psi^I\rho e^{i\zeta}/H + \bar{\psi}^I\rho e^{-i\zeta}/H = 2l^I(u)\ , \tag{5.65}$$

where $l^I(u)$ are arbitrary real functions of u obeying the constraint $l^I\xi_I = 0$ due to (5.58), and the factor 2 was chosen for later convenience. By virtue of (5.54), $\mathcal{N}_{IJ}$ is likewise independent of $w + \bar{w}$, and hence (4.54) implies

$$\bar{\mathcal{N}}_{IJ}\psi^J\rho e^{i\zeta}/H + \mathcal{N}_{IJ}\bar{\psi}^J\rho e^{-i\zeta}/H = 2m_I(u)\ , \tag{5.66}$$

with $m_I(u)$ again some real functions. Since $(\mathrm{Im}\,\mathcal{N})_{IJ}$ is invertible, (5.65) together with (5.66) give an expression for ψ^I in terms of l^I and m_I,

$$\psi^I\rho e^{i\zeta}/H = l^I(u) + i(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\left(m_J(u) - (\mathrm{Re}\,\mathcal{N})_{JK}l^K(u)\right)\ . \tag{5.67}$$

Reality of ψ yields in addition

$$\bar{X}\cdot\xi\left(F_Il^I - m_IX^I\right) = X\cdot\xi\left(\bar{F}_Il^I - m_I\bar{X}^I\right)\ . \tag{5.68}$$

In what follows, we shall solve the above equations for the simple model with prepotential $F = -iZ^0Z^1$ introduced in section 4.3.2. In this case one has

$$X\cdot\xi = \frac{\xi_0 + \xi_1 z}{\sqrt{2(z+\bar{z})}}\ . \tag{5.69}$$

A sufficient condition for (5.68) to be satisfied is $l^I = 0$ and $z = \bar{z}$. Then also the constraint $l^I \xi_I = 0$ is met, and we obtain

$$\psi^0 = -\frac{im_0}{z} H^{3/2} \ , \qquad \psi^1 = -im_1 z H^{3/2} \ , \qquad \psi = -H^{3/2} \frac{m_1 z + m_0}{\sqrt{z}} \ . \tag{5.70}$$

(4.46) and (5.52) reduce respectively to

$$\partial_w z = ig \left(\frac{2z}{H} \right)^{1/2} (-\xi_0 + \xi_1 z) \ , \qquad \partial_w \sqrt{H} = ig \frac{\xi_0 + \xi_1 z}{\sqrt{2z}} \ . \tag{5.71}$$

Making use of this, one finds that (4.55) holds identically. Moreover, the eqns. (5.71) imply that H is again given by (4.83), where we choose $f(u) = 1$. Let us further assume that the m_I are constants and z is independent of u. z is then a function of y only, where we defined $w = x + iy$. With these choices, the conditions (5.64) are fulfilled as well. To determine the wave profile $\mathcal{G}$, one first observes that (5.59) leads to

$$\mathcal{G} = \mathcal{G}_1(u, x) + \mathcal{G}_2(u, y) \ . \tag{5.72}$$

From (5.62) it is clear that $\partial_w^2 \mathcal{G}$ does not depend on x, and thus

$$\mathcal{G}_1(u, x) = n(u) x^2 + h(u) x + j(u) \ , \tag{5.73}$$

for some functions $n(u), h(u), j(u)$. By shifting the coordinate v, we can always set $j = 0$ without loss of generality. Integrating once the Einstein equation (5.63) yields

$$\partial_y \mathcal{G}_2 = H \left[\frac{\sqrt{2}}{g} \left(-\frac{m_0^2}{z} + m_1^2 z \right) - \frac{n(u)}{\sqrt{2} g \xi_1 (-\xi_0 + \xi_1 z)} + k(u) \right] \ , \tag{5.74}$$

with $k(u)$ arbitrary at this stage. Compatibility of this with (5.62) requires

$$k(u) = \frac{\sqrt{2} m_1}{g \xi_1} (\xi_0 m_1 - \xi_1 m_0) \ , \qquad n(u) = -2(\xi_0 m_1 - \xi_1 m_0)^2 \ . \tag{5.75}$$

Before integrating (5.74) again, we explicitely solve the Killing spinor equations (5.38)-(5.41). From the equations for $\partial_w b$ and $\partial_{\bar{w}} b$ one obtains $b = \beta(u)/\sqrt{H}$, where $\beta(u)$ denotes an arbitrary function. Then, (5.61) gives

$$a - \bar{c} = \frac{1}{gz} (-\xi_0 + \xi_1 z)(m_1 z - m_0) \beta(u) \ , \tag{5.76}$$

Deriving this with respect to u and using the relations (5.38), one gets

$$\frac{\beta'(u)}{\beta(u)} = -\frac{in(u)}{\sqrt{2} \xi_1} \ , \tag{5.77}$$

together with $k(u) = 0$, i.e.,

$$m_1 = 0 \qquad \text{or} \qquad \xi_0 m_1 = \xi_1 m_0 \ . \tag{5.78}$$

We shall choose here the latter possibility. Then $n(u) = 0$, and (5.77) implies that β must be constant, $\beta = \beta_0$. The remaining Killing spinor equations can then be easily integrated, with the result

$$\begin{aligned}
a &= \frac{\beta_0}{2g}\left(m_1\xi_1 z + \frac{m_0\xi_0}{z}\right) + \alpha(u) \ , \\
c &= -\frac{\bar\beta_0}{2g}\left(m_1\xi_1 z + \frac{m_0\xi_0}{z}\right) + \bar\alpha(u) + \frac{2\bar\beta_0}{g}\xi_0 m_1 \ , \\
b &= \bar d = \frac{\beta_0}{\sqrt{H}} \ ,
\end{aligned} \tag{5.79}$$

where $\alpha(u)$ obeys

$$\alpha'(u) = -\frac{\beta_0}{2}h(u) \ . \tag{5.80}$$

The first Killing spinor is recovered for $\beta_0 = 0$. The Killing vector associated to (5.79) has components

$$\begin{aligned}
V_+ &= 2\sqrt{2}\frac{|\beta_0|^2}{H} \ , \\
V_- &= -\frac{|\beta_0|^2}{\sqrt{2}g^2}\left(m_1\xi_1 z + \frac{m_0\xi_0}{z} - 2\xi_0 m_1\right)^2 - 2\sqrt{2}|\alpha + \frac{\beta_0}{g}\xi_0 m_1|^2 \ , \\
V_\bullet &= V_{\bar\bullet} = -\sqrt{\frac{2}{H}}\left(\alpha\bar\beta_0 + \bar\alpha\beta_0 + \frac{2|\beta_0|^2}{g}\xi_0 m_1\right) \ ,
\end{aligned} \tag{5.81}$$

and norm squared

$$V^2 = -\frac{4|\beta_0|^4}{g^2}z\left(m_1 - \frac{m_0}{z}\right)^2 - \frac{16}{H}\mathrm{Im}^2\left[\bar\beta_0\left(\alpha + \frac{\beta_0}{g}\xi_0 m_1\right)\right] \ , \tag{5.82}$$

which is in general negative, unless $\beta_0 = 0$, so that the solution belongs to the timelike class as well. This explains also why the uu component of the Einstein equations is implied by the integrability conditions (5.57).

Finally, (5.74) yields the wave profile

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 = h(u)x - \frac{m_1^2}{4\xi_1^2 g^2}H^2 \ , \tag{5.83}$$

where

$$H = 4\xi_0\xi_1\sinh^2\sqrt{2}gy \ . \tag{5.84}$$

The scalar field z and metric are given by (4.84) and (4.85) respectively, and the fluxes read

$$F^0 = \frac{2m_0}{z} du \wedge dy \ , \qquad F^1 = 2m_1 z du \wedge dy \ . \tag{5.85}$$

Note that for $y \to 0$, when the scalar goes to its critical value, we have $H \to 8\xi_0\xi_1 g^2 y^2$, and thus (if we choose $h(u) = 0$) the wave profile becomes

$$\mathcal{G} = -16 m_1^2 \xi_0^2 g^2 y^4 \ , \tag{5.86}$$

which means that the solution reduces to a subclass of the charged generalization of the Kaigorodov spacetime found in [45].

This concludes the explicit example of a half-supersymmetric background with $d\chi = 0$, $\psi \neq 0$.

Next we have to consider the case when $\psi = 0$. Then, (4.44) and (4.55) give

$$\psi^I \xi_I = A^I \xi_I = F^I \xi_I = 0 \ . \tag{5.87}$$

Contracting (5.46) with ξ_I, taking into account that $\psi = \psi^I \xi_I = 0$ and assuming as before that $g^{\alpha\bar{\beta}} \mathcal{D}_\alpha X^I \mathcal{D}_{\bar{\beta}} \bar{X}^J \xi_I \xi_J \neq 0$ (otherwise, as was explained above, the scalar fields would be constant), we get

$$\frac{a - \bar{c}}{b} + \frac{\bar{a} - c}{d} = 0 \ . \tag{5.88}$$

Plugging this back into (5.46), one obtains $\psi^I = 0$, so there are no fluxes turned on in this case. The Killing spinor equations together with the integrability conditions (5.47) imply

$$\partial_\mu (\bar{b} X^I - d \bar{X}^I) \xi_I = 0 \ , \tag{5.89}$$

hence

$$(\bar{b} X^I - d \bar{X}^I) \xi_I = \lambda \ , \tag{5.90}$$

with λ a constant.

Let us first assume that $b \bar{X}\cdot\xi + \bar{d} X\cdot\xi = 0$, so that $X\cdot\xi\, \bar{X}\cdot\xi$ is constant due to (5.45). Then, the Killing spinor equations for b and d simplify to

$$\begin{aligned}
\partial_u b &= -iA_u b + \frac{ig\sqrt{2}}{H} X\cdot\xi (a - \bar{c}) = iA_u b - \frac{ig\sqrt{2}}{H} X\cdot\xi (a - \bar{c}) - b \partial_u \ln \frac{\bar{X}\cdot\xi}{X\cdot\xi} \ , \\
\partial_w b &= -iA_w b + ig\sqrt{2} \bar{X}\cdot\xi E^{\bullet}{}_w b = iA_w b - ig\sqrt{2} \bar{X}\cdot\xi E^{\bullet}{}_w b - b \partial_w \ln \frac{\bar{X}\cdot\xi}{X\cdot\xi} \ , \\
\partial_{\bar{w}} b &= -iA_{\bar{w}} b + ig\sqrt{2} X\cdot\xi E^{\bar{\bullet}}{}_{\bar{w}} b = iA_{\bar{w}} b - ig\sqrt{2} X\cdot\xi E^{\bar{\bullet}}{}_{\bar{w}} b - b \partial_{\bar{w}} \ln \frac{\bar{X}\cdot\xi}{X\cdot\xi} \ ,
\end{aligned}$$

from which one obtains

$$A_u = \frac{i}{2}\partial_u \ln \frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi} + \frac{g\sqrt{2}}{H} X{\cdot}\xi \frac{a - \bar{c}}{b} \ , \tag{5.91}$$

$$A_w = \frac{i}{2}\partial_w \ln \frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi} + g\sqrt{2}\bar{X}{\cdot}\xi E^{\bullet}{}_w \ , \qquad A_{\bar{w}} = \frac{i}{2}\partial_{\bar{w}} \ln \frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi} + g\sqrt{2}X{\cdot}\xi E^{\bar{\bullet}}{}_{\bar{w}} \ , \tag{5.92}$$

as well as

$$b = \lambda_1 \sqrt{\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}} \ , \qquad d = -\bar{\lambda}_1 \sqrt{\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}} \ ,$$

where $\lambda_1 \neq 0$ is an integration constant. Using the expression (5.92) for $A_{\bar{w}}$ in (4.50) leads to

$$E^{\bullet}{}_w = \sqrt{\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}} f(u,w) \ , \tag{5.93}$$

with $f(u,w)$ an arbitrary function that, as before, can be set to unity without loss of generality. Then we have

$$\rho = 1 \ , \qquad e^{i\zeta} = \sqrt{\frac{X{\cdot}\xi}{\bar{X}{\cdot}\xi}} \ ,$$

and thus

$$A_u + \partial_u \zeta = \frac{g\sqrt{2}}{\lambda_1 H} \sqrt{X{\cdot}\xi\, \bar{X}{\cdot}\xi}(a - \bar{c}) \ ,$$

where we used (5.91). This, together with (4.47) gives the relation

$$c = \bar{a} + \frac{i\bar{\lambda}_1}{2g\sqrt{2X{\cdot}\xi\, \bar{X}{\cdot}\xi}} (E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) \tag{5.94}$$

between $\bar{a}$ and c, that can be substituted into the Killing spinor equations for a and c,

which become

$$
\begin{aligned}
\frac{\partial_u a}{\lambda_1} &= 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}E^{+}{}_u - (E^{+}{}_{u,w} - E^{+}{}_{w,u}) \\
&= 2ig\sqrt{2X\cdot\xi\bar{X}\cdot\xi}E^{+}{}_u + (E^{+}{}_{u,\bar{w}} - E^{+}{}_{\bar{w},u}) + \frac{i}{2g\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}}\partial_u(E^{+}{}_{w,\bar{w}} - E^{+}{}_{\bar{w},w}) \ , \\
\frac{\partial_v a}{\lambda_1} &= -8ig\sqrt{X\cdot\xi\,\bar{X}\cdot\xi} \ , \\
\frac{\partial_w a}{\lambda_1} &= 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}E^{+}{}_w \\
&= 2ig\sqrt{2X\cdot\xi\bar{X}\cdot\xi}E^{+}{}_w + (E^{+}{}_{w,\bar{w}} - E^{+}{}_{\bar{w},w}) + \frac{i}{2g\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}}\partial_w(E^{+}{}_{w,\bar{w}} - E^{+}{}_{\bar{w},w}) \ , \\
\frac{\partial_{\bar{w}} a}{\lambda_1} &= 2ig\sqrt{2X\cdot\xi\bar{X}\cdot\xi}E^{+}{}_{\bar{w}} + \frac{i}{2g\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}}\partial_{\bar{w}}(E^{+}{}_{w,\bar{w}} - E^{+}{}_{\bar{w},w}) \\
&= 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}E^{+}{}_{\bar{w}} + (E^{+}{}_{w,\bar{w}} - E^{+}{}_{\bar{w},w}) \ ,
\end{aligned}
\tag{5.95}
$$

so that E^+ is constrained by

$$
\begin{aligned}
\partial_u(E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) &= 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(E^{+}{}_{w,u} - E^{+}{}_{u,w} + E^{+}{}_{\bar{w},u} - E^{+}{}_{u,\bar{w}}) \ , \\
\partial_w(E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) &= 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) \ , \\
\partial_{\bar{w}}(E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) &= -2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}}) \ .
\end{aligned}
\tag{5.96}
$$

Integrating the last two equations, one has

$$
E^{+}{}_{\bar{w},w} - E^{+}{}_{w,\bar{w}} = \mathcal{E}(u)\exp\left[2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(w-\bar{w})\right] \ , \tag{5.97}
$$

with $\mathcal{E}(u)$ some imaginary function. This implies

$$
E^{+}{}_w - \frac{\mathcal{E}(u)}{4ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}}\exp\left[2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(w-\bar{w})\right] = \partial_w m \tag{5.98}
$$

for some real function m. By shifting v and $\mathcal{G}$ appropriately, we can thus set $E^{+}{}_w = E^{+}{}_{\bar{w}} = 0$. Then, (5.96) gives

$$
(\partial_w + \partial_{\bar{w}})E^{+}{}_u = 0 \ , \tag{5.99}
$$

while from (5.94) one obtains $c = \bar{a}$. The Killing spinor equations (5.95) are easily integrated, with the result

$$
a = -8ig\sqrt{X\cdot\xi\,\bar{X}\cdot\xi}\lambda_1 v + \lambda_1\alpha(u) \ , \tag{5.100}
$$

where α satisfies

$$
\alpha'(u) = 2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}E^{+}{}_u - \partial_w E^{+}{}_u \ . \tag{5.101}
$$

The latter relation determines ${E^+}_u$,

$$ {E^+}_u = \frac{\alpha'(u)}{2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}} + \hat{\mathcal{E}}(u)\exp\left[2ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(w-\bar{w})\right] , \tag{5.102} $$

with $\hat{\mathcal{E}}(u)$ real and otherwise arbitrary. Note that ${E^+}_u$ is independent of $w+\bar{w}$. Eqns. (4.46), (5.42) and (5.43) boil down to $\partial_u z^\alpha = 0$ and

$$ \partial_w z^\alpha = \partial_{\bar{w}} z^\alpha = ig\sqrt{2}g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}^I\xi_I\sqrt{\frac{X\cdot\xi}{\bar{X}\cdot\xi}} , \tag{5.103} $$

so that the scalar fields are functions of $x = (w+\bar{w})/2$ only. The uu component of the Einstein equations reads

$$ \partial_w\partial_{\bar{w}}{E^+}_u + ig\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}(\partial_w - \partial_{\bar{w}}){E^+}_u = 2g^2\left[(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\xi_I\xi_J + 4X\cdot\xi\,\bar{X}\cdot\xi\right]{E^+}_u . $$

While the lhs is independent of x, the prefactor of ${E^+}_u$ on the rhs depends in general nontrivially on x. This is compatible only if ${E^+}_u = 0$, hence $\alpha(u)$ is constant due to (5.101). The function H appearing in the metric follows from (4.48), yielding

$$ H = h(u)\exp\left[-4g\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}\,y\right] , \tag{5.104} $$

where $y = (w-\bar{w})/2i$, and we can always choose $h(u) = -2\sqrt{2}$ by redefining the coordinate u.

In conclusion, the metric is given by

$$ ds^2 = 2\left\{\exp\left[4g\sqrt{2X\cdot\xi\,\bar{X}\cdot\xi}\,y\right]dudv + dy^2 + dx^2\right\} , \tag{5.105} $$

which is simply $\mathrm{AdS}_3\times\mathbb{R}$. The dependence of the scalars on the $\mathbb{R}$-coordinate x is governed by (5.103), that can be rewritten as

$$ \frac{dz^\alpha}{dx} = 2ig\sqrt{2\mathcal{C}}g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}\ln(\bar{X}\cdot\xi e^{\mathcal{K}/2}) , \tag{5.106} $$

where the constant $\mathcal{C}$ is defined by

$$ \mathcal{C} = X\cdot\xi\,\bar{X}\cdot\xi . $$

The solution to the Killing spinor equations reads

$$ a = \bar{c} = -8ig\sqrt{X\cdot\xi\,\bar{X}\cdot\xi}\lambda_1 v + \lambda_1\alpha , \qquad b = \lambda_1\sqrt{\frac{X\cdot\xi}{\bar{X}\cdot\xi}} , \qquad d = -\bar{\lambda}_1\sqrt{\frac{X\cdot\xi}{\bar{X}\cdot\xi}} , \tag{5.107} $$

which reduces to the first covariantly constant spinor if we rescale $\alpha \to \alpha/\lambda_1$ and then take $\lambda_1 \to 0$. The Killing vector constructed from (5.107) has components

$$V_+ = 2\sqrt{2}|\lambda_1|^2 \,, \qquad V_- = -2\sqrt{2}|-8ig\sqrt{X\cdot\xi\,\bar{X}\cdot\xi}\lambda_1 v + \lambda_1\alpha|^2 \,,$$
$$V_\bullet = 2|\lambda_1|^2[16ig\bar{X}\cdot\xi v + \sqrt{\frac{\bar{X}\cdot\xi}{X\cdot\xi}}(\bar{\alpha}-\alpha)] \,, \qquad V_{\bar{\bullet}} = \overline{V_\bullet} \,, \tag{5.108}$$

and norm squared

$$V^2 = -4|\lambda_1|^4(\bar{\alpha}+\alpha)^2 \,, \tag{5.109}$$

which is negative unless $\lambda_1 = 0$ or $\mathrm{Re}\,\alpha = 0$, so in general the solution again belongs also to the timelike class.

The final case to consider is $\psi = 0$, $b\bar{X}\cdot\xi + \bar{d}X\cdot\xi \neq 0$. Then, equ. (5.47) together with (5.90) implies that

$$A_\mu = -i\frac{b\partial_\mu\bar{X}\cdot\xi - \bar{d}\partial_\mu X\cdot\xi}{b\bar{X}\cdot\xi + \bar{d}X\cdot\xi} \tag{5.110}$$
$$= \left(1 - \frac{\bar{\lambda}}{2\bar{X}\cdot\xi b}\right)^{-1}\left(i\partial_\mu \ln\sqrt{\frac{X\cdot\xi}{\bar{X}\cdot\xi}} - i\frac{\bar{\lambda}}{\bar{X}\cdot\xi b}\partial_\mu \ln\sqrt{X\cdot\xi}\right) \,.$$

One easily shows that $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$, and thus $A_\mu = \partial_\mu\varsigma$ for some v-independent function ς. Using this and (4.48), we can integrate (4.50) to obtain

$$E^\bullet{}_w = e^{-i\alpha}H^{-1/2}f(u,w) \,, \tag{5.111}$$

where $f(u,w)$ denotes an arbitrary function that, as was explained before, can be set to unity without loosing generality. Then one has $\rho = H^{-1/2}$ and $\zeta = -\varsigma$, and (4.47) yields $E^+{}_w = E^+{}_{\bar{w}} = 0$. The Killing spinor equations for b reduce to

$$\partial_u b = -ib\partial_u\varsigma + \frac{ig\sqrt{2}}{H}X\cdot\xi(a-\bar{c}) \,,$$
$$\partial_w b = -b\partial_w\left(\frac{1}{2}\ln H + i\varsigma\right) + \frac{ig\bar{\lambda}\sqrt{2}}{\sqrt{H}}e^{-i\varsigma} \,,$$
$$\partial_{\bar{w}} \ln b = -\partial_{\bar{w}}\left(\frac{1}{2}\ln H + i\varsigma\right) \,, \tag{5.112}$$

from which we get

$$b = (ig\bar{\lambda}\sqrt{2}w + \hat{b}(u))e^{-i\varsigma}H^{-1/2} \,, \tag{5.113}$$

with $\hat{b}$ obeying

$$\partial_u\hat{b} = ig\sqrt{2}X\cdot\xi(a-\bar{c})e^{i\varsigma}H^{-1/2} + \frac{1}{2}(ig\bar{\lambda}\sqrt{2}w + \hat{b})\partial_u \ln H \,. \tag{5.114}$$

It follows from the Killing spinor equations for a and c that

$$\partial_\mu c = \frac{d}{b}\partial_\mu a \ . \tag{5.115}$$

Since (5.90) combined with $\bar{b}b = \bar{d}d$ leads to

$$\frac{\lambda}{\bar{\lambda}} = -\frac{d}{b} \ , \tag{5.116}$$

one obtains[13]

$$c = -\frac{\lambda}{\bar{\lambda}}a + \kappa \ , \tag{5.117}$$

where κ is a constant that satisfies $\bar{\lambda}\kappa = \lambda\bar{\kappa}$ due to (5.88). The Killing spinor equations for a boil down to

$$\begin{aligned}
\partial_u a &= ig\bar{\lambda}\sqrt{2}E^+{}_u - (ig\bar{\lambda}\sqrt{2}w + \hat{b})\partial_w E^+{}_u \ , \\
\partial_v a &= -4ig\bar{\lambda} \ , \\
\partial_w a &= 0 \ , \\
\partial_{\bar{w}} a &= -\partial_u \hat{b} \ ,
\end{aligned}$$

so that

$$a = -4ig\bar{\lambda}v - \bar{w}\partial_u\hat{b} + \hat{a}(u) \ , \tag{5.118}$$

where $\hat{a}$ satisfies

$$\partial_u\hat{a} = \bar{w}\partial_u^2\hat{b} + ig\bar{\lambda}\sqrt{2}E^+{}_u - (ig\bar{\lambda}\sqrt{2}w + \hat{b})\partial_w E^+{}_u \ . \tag{5.119}$$

(5.114) becomes

$$\begin{aligned}
\partial_u\hat{b} = \ &\frac{1}{2\lambda}(\lambda\bar{w}\partial_u\hat{b} + \bar{\lambda}w\partial_u\bar{\hat{b}} - \lambda\hat{a} - \bar{\lambda}\bar{\hat{a}} + \bar{\lambda}\kappa)\partial_{\bar{w}}\ln H \\
&+\frac{1}{2}(ig\bar{\lambda}\sqrt{2}w + \hat{b})\partial_u\ln H \ .
\end{aligned}$$

Deriving (5.119) with respect to w we obtain $\partial_w^2 E^+{}_u = 0$, hence

$$E^+{}_u = \omega_1(u)w\bar{w} + \omega_2(u)w + \bar{\omega}_2(u)\bar{w} + \omega_3(u) \ , \tag{5.120}$$

where ω_1 and ω_3 are real. Shifting the coordinate v one can set $\omega_3 = 0$ without loss of generality. Plugging back this expression for $E^+{}_u$ into (5.119) one gets

$$\partial_u\hat{a} = -\omega_2\hat{b} \ , \qquad \partial_u^2\hat{b} = \omega_1\hat{b} - ig\sqrt{2}\bar{\lambda}\bar{\omega}_2 \ . \tag{5.121}$$

[13]Here we assume $\lambda \neq 0$. The case $\lambda = 0$, i.e. $\bar{b}X\cdot\xi = d\bar{X}\cdot\xi$, was already considered earlier.

Note that $E^+{}_u$ must in addition satisfy the uu component of the Einstein equations, namely

$$\begin{aligned}\partial_w\partial_{\bar{w}}E^+{}_u &= 2g\sqrt{\frac{2}{H}}\mathrm{Im}\left(X{\cdot}\xi e^{i\varsigma}\partial_w E^+{}_u\right) + \frac{1}{2}\partial_u^2\ln H + \frac{1}{4}\left(\partial_u \ln H\right)^2 \\ &+ \frac{2ig\sqrt{2}\partial_u(X{\cdot}\xi\,\bar{X}{\cdot}\xi)\mathrm{Re}\left(\lambda\bar{w}\partial_u\hat{b} - \lambda\hat{a} + \frac{\bar{\lambda}\kappa}{2}\right)}{2\sqrt{H}\bar{\lambda}X{\cdot}\xi\left(\bar{\hat{b}} - ig\lambda\sqrt{2}\bar{w}\right)e^{i\varsigma} - H|\lambda|^2} \ .\end{aligned} \tag{5.122}$$

Solving these equations in general seems to be difficult. A simplification can be made by assuming $a = \bar{c}$, which happens for $\partial_u\hat{b} = \partial_u\hat{a} = 0$, $\lambda\hat{a} + \bar{\lambda}\bar{\hat{a}} = \lambda\bar{\kappa}$. If we take in addition $\omega_1 = \omega_2 = \omega_3 = 0$ (and thus $E^+{}_u = 0$), (5.119) is satisfied. Moreover, from (5.114) one gets $\partial_u H = 0$, and (5.42) yields $\partial_u z^\alpha = 0$. Note that $\hat{b}$ can be set to zero by a constant shift of w, cf. (5.113). Using (5.116), the equations (4.46) and (5.43) simplify to

$$\begin{aligned}\partial_w z^\alpha &= ig\sqrt{2}\xi_I e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I g^{\alpha\bar{\beta}}e^{-i\varsigma}H^{-1/2} \ , \\ \partial_{\bar{w}} z^\alpha &= -ig\sqrt{2}\xi_I e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I g^{\alpha\bar{\beta}}\frac{w}{\bar{w}}e^{-i\varsigma}H^{-1/2} \ ,\end{aligned} \tag{5.123}$$

which imply

$$(w\partial_w + \bar{w}\partial_{\bar{w}})z^\alpha = 0 \ , \tag{5.124}$$

i.e., $\partial_r z^\alpha = 0$, where we introduced polar coordinates r, θ according to $w = re^{i\theta}$. The scalar fields depend thus on the angular coordinate θ only. This, in turn, gives $A_u = A_r = 0$ and $\varsigma = \varsigma(\theta)$. By means of the separation ansatz $\sqrt{H} = rh(\theta)$, (4.48) becomes

$$h(\theta) - ih'(\theta) = 2\sqrt{2}ig\bar{X}{\cdot}\xi e^{i(\theta-\varsigma)} \ , \tag{5.125}$$

and the flow equation reduces to

$$\frac{dz^\alpha}{d\theta} = -2\sqrt{2}g\xi_I e^{\mathcal{K}/2}\mathcal{D}_{\bar{\beta}}\bar{Z}^I g^{\alpha\bar{\beta}}e^{i(\theta-\varsigma)}h^{-1} \ . \tag{5.126}$$

(5.110), with $A_\theta = \partial_\theta\varsigma$, can be rewritten as

$$e^{i\theta}\partial_\theta(\bar{X}{\cdot}\xi e^{-i\varsigma}) = e^{-i\theta}\partial_\theta(X{\cdot}\xi e^{i\varsigma}) \ . \tag{5.127}$$

Solving (5.125) for $X{\cdot}\xi e^{i\varsigma}$ and plugging the result into (5.127), one finds that (5.127) holds identically. In conclusion, the unknown functions z^α, h and ς are determined by the system of ordinary differential equations (5.125) and (5.126). In the following, we

shall solve these equations for the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ model with prepotential $F = (Z^1)^3/Z^0$. Choosing $Z^0 = 1$, $Z^1 = -z$, the symplectic vector reads

$$v = \begin{pmatrix} 1 \\ -z \\ z^3 \\ 3z^2 \end{pmatrix} . \tag{5.128}$$

The Kähler potential and metric are given respectively by

$$e^{-\mathcal{K}} = 8(\mathrm{Im}z)^3 , \qquad g_{z\bar{z}} = -\frac{3}{(z-\bar{z})^2} , \tag{5.129}$$

so we must have $\mathrm{Im}z > 0$. For the scalar potential one obtains

$$V = g^2 V_3 = -\frac{4g^2\xi_1^2}{3\mathrm{Im}z} . \tag{5.130}$$

Notice that this model permits to introduce a gauging without having a scalar potential, by choosing $\xi_1 = 0$, $\xi_0 \neq 0$.

Solving (5.125) for $e^{i(\theta-\varsigma)}$ and plugging into (5.126) gives in general

$$dz^\alpha = ig^{\alpha\bar{\beta}}\partial_{\bar{\beta}} \ln(\bar{X}\cdot\xi e^{\mathcal{K}/2})(d\theta - id\ln h) , \tag{5.131}$$

which reduces to

$$dz = -i(z-\bar{z})(d\theta - id\ln h) \tag{5.132}$$

for the model under consideration, if we make the choice $\xi_1 = 0$. Subtracting this from its complex conjugate yields

$$h^2 = \frac{A}{\mathrm{Im}z} , \tag{5.133}$$

with A a real positive constant. (5.125) implies

$$h^2 + h'^2 = 8g^2 X\cdot\xi\,\bar{X}\cdot\xi , \tag{5.134}$$

which can be easily integrated to give

$$\mathrm{Im}z = \frac{g\xi_0}{\sqrt{A}}\sin 2\theta . \tag{5.135}$$

Positivity of $\mathrm{Im}z$ restricts θ to the range $0 < \theta < \pi/2$. Using (5.135) in the sum of (5.132) and its complex conjugate allows to determine also $\mathrm{Re}z$. Eventually this leads to

$$z = z_0 - \frac{g\xi_0}{\sqrt{A}}e^{-2i\theta} , \tag{5.136}$$

where z_0 denotes a real constant. Then, (5.133) yields

$$h^2(\theta) = \frac{A^{3/2}}{g\xi_0 \sin 2\theta} \,, \tag{5.137}$$

so that

$$H = \frac{r^2 A^{3/2}}{g\xi_0 \sin 2\theta} \,. \tag{5.138}$$

Finally, (5.125) determines $\varsigma = 3\theta$. The metric becomes

$$ds^2 = \frac{2g\xi_0 \sin 2\theta}{A^{3/2}} \left[-\frac{2\sqrt{2} du dv}{r^2} + \frac{dr^2}{r^2} + d\theta^2 \right] \,, \tag{5.139}$$

and thus the spacetime is conformal to AdS_3 times an interval.

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A. Conventions

We use the notations and conventions of [41], which are briefly summarized here. More details can be found in appendix A of [41].

The signature is mostly plus. Late greek letters $\mu, \nu, \ldots$ are curved spacetime indices, while early latin letters $a, b, \ldots = 0, \ldots, 3$ and $A, B, \ldots = +, -, \bullet, \bar{\bullet}$ refer to the corresponding tangent space, cf. also appendix B.

Self-dual and anti-self-dual field strengths are defined by

$$F^{\pm I}_{ab} = \frac{1}{2}(F^I_{ab} \pm \tilde{F}^I_{ab}) \,, \qquad \tilde{F}^I_{ab} \equiv -\frac{i}{2}\epsilon_{abcd} F^{Icd} \,, \tag{A.1}$$

where $\epsilon_{0123} = 1$, $\epsilon^{0123} = -1$. We also introduce

$$\epsilon^{\mu\nu\rho\sigma} = e\, e^\mu_a e^\nu_b e^\rho_c e^\sigma_d \epsilon^{abcd} \,. \tag{A.2}$$

The p-form associated to an antisymmetric tensor $T_{\mu_1\dots\mu_p}$ is

$$T = \frac{1}{p!} T_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} , \tag{A.3}$$

and the exterior derivative acts as[14]

$$dT = \frac{1}{p!} T_{\mu_1\dots\mu_p,\nu} dx^{\nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} . \tag{A.4}$$

Antisymmetric tensors are often contracted with Γ-matrices as in $\Gamma \cdot F \equiv \Gamma^{ab} F_{ab}$. Moreover, we defined $X{\cdot}\xi \equiv X^I \xi_I$.

$i, j, \dots = 1, 2$ are SU(2) indices, whose raising and lowering is done by complex conjugation. The Levi-Civita ϵ^{ij} has the property

$$\epsilon_{ij} \epsilon^{jk} = -\delta_i{}^k , \tag{A.5}$$

where in principle ϵ^{ij} is the complex conjugate of ϵ_{ij}, but we can choose $\epsilon = i\sigma_2$, such that

$$\epsilon_{12} = \epsilon^{12} = 1 . \tag{A.6}$$

The Pauli matrices $\sigma_{xi}{}^j$ $(x = 1, 2, 3)$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.7}$$

They allow to switch from SU(2) indices to vector quantities using the convention

$$A_i{}^j \equiv i\vec{A} \cdot \vec{\sigma}_i{}^j . \tag{A.8}$$

At various places in the main text we use σ-matrices with only lower or upper indices, defined by

$$\vec{\sigma}_{ij} \equiv \vec{\sigma}_i{}^k \epsilon_{kj} , \qquad i\vec{\sigma}^{ij} = (i\vec{\sigma}_{ij})^* . \tag{A.9}$$

Notice that both $\vec{\sigma}_{ij}$ and $\vec{\sigma}^{ij}$ are symmetric.

Spinors carrying an index i are chiral, e.g. for the supersymmetry parameter one has

$$\Gamma_5 \epsilon^i = \epsilon^i , \qquad \Gamma_5 \epsilon_i = -\epsilon_i , \tag{A.10}$$

and the same holds for the gravitino ψ^i_μ. Note however that for some spinors, the upper index denotes negative chirality rather than positive chirality, for instance the gauginos obey

$$\Gamma_5 \lambda^{\alpha i} = -\lambda^{\alpha i} , \qquad \Gamma_5 \lambda^\alpha_i = \lambda^\alpha_i , \tag{A.11}$$

[14] Our definitions for p-forms, equ. (A.3), and for exterior derivatives, equ. (A.4), are the only points where our conventions differ from those of [41].

as is also evident from the supersymmetry transformations. The charge conjugate of a spinor χ is

$$\chi^C = \Gamma_0 C^{-1} \chi^* \,, \tag{A.12}$$

with the charge conjugation matrix C. Majorana spinors are defined by $\chi = \chi^C$, and chiral spinors obey $\chi_i^C = \chi^i$.

B. Spinors and forms

In this appendix, we summarize the essential information needed to realize the spinors of Spin(3,1) in terms of forms. For more details, we refer to [46]. Let $V = \mathbb{R}^{3,1}$ be a real vector space equipped with the Lorentzian inner product $\langle \cdot, \cdot \rangle$. Introduce an orthonormal basis e_1, e_2, e_3, e_0, where e_0 is along the time direction, and consider the subspace U spanned by the first two basis vectors e_1, e_2. The space of Dirac spinors is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$, with basis $1, e_1, e_2, e_{12} = e_1 \wedge e_2$. The gamma matrices are represented on Δ_c as

$$\begin{aligned} \Gamma_0 \eta &= -e_2 \wedge \eta + e_2 \rfloor \eta \,, \qquad & \Gamma_1 \eta &= e_1 \wedge \eta + e_1 \rfloor \eta \,, \\ \Gamma_2 \eta &= e_2 \wedge \eta + e_2 \rfloor \eta \,, \qquad & \Gamma_3 \eta &= i e_1 \wedge \eta - i e_1 \rfloor \eta \,, \end{aligned} \tag{B.1}$$

where

$$\eta = \frac{1}{k!} \eta_{j_1 \ldots j_k} e_{j_1} \wedge \ldots \wedge e_{j_k}$$

is a k-form and

$$e_i \rfloor \eta = \frac{1}{(k-1)!} \eta_{i j_1 \ldots j_{k-1}} e_{j_1} \wedge \ldots \wedge e_{j_{k-1}} \,.$$

One easily checks that this representation of the gamma matrices satisfies the Clifford algebra relations $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$. The parity matrix is defined by $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$, and one finds that the even forms $1, e_{12}$ have positive chirality, $\Gamma_5 \eta = \eta$, while the odd forms e_1, e_2 have negative chirality, $\Gamma_5 \eta = -\eta$, so that Δ_c decomposes into two complex chiral Weyl representations $\Delta_c^+ = \Lambda^{\mathrm{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\mathrm{odd}}(U \otimes \mathbb{C})$. Note that Spin(3,1) is isomorphic to $\mathrm{SL}(2, \mathbb{C})$, which acts with the fundamental representation on the positive chirality Weyl spinors.

Let us define the auxiliary inner product

$$\langle \sum_{i=1}^{2} \alpha_i e_i, \sum_{j=1}^{2} \beta_j e_j \rangle = \sum_{i=1}^{2} \alpha_i^* \beta_i \tag{B.2}$$

on $U \otimes \mathbb{C}$, and then extend it to Δ_c. The Spin(3,1) invariant Dirac inner product is then given by

$$D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle \,. \tag{B.3}$$

The Majorana inner product that we use is[15]

$$A(\eta,\theta) = \langle C\eta^*, \theta\rangle\,, \tag{B.4}$$

with the charge conjugation matrix $C = \Gamma_{12}$. Using the identities

$$\Gamma_a^* = -C\Gamma_0\Gamma_a\Gamma_0 C^{-1}\,, \qquad \Gamma_a^T = -C\Gamma_a C^{-1}\,, \tag{B.5}$$

it is easy to show that (B.4) is Spin(3,1) invariant as well.

The charge conjugation matrix C acts on the basis elements as

$$C1 = e_{12}\,, \quad Ce_{12} = -1\,, \quad Ce_1 = -e_2\,, \quad Ce_2 = e_1\,. \tag{B.6}$$

In many applications it is convenient to use a basis in which the gamma matrices act like creation and annihilation operators, given by

$$\begin{aligned}
\Gamma_+\eta &\equiv \frac{1}{\sqrt{2}}(\Gamma_2+\Gamma_0)\,\eta = \sqrt{2}\,e_2\rfloor\eta\,, \qquad & \Gamma_-\eta &\equiv \frac{1}{\sqrt{2}}(\Gamma_2-\Gamma_0)\,\eta = \sqrt{2}\,e_2\wedge\eta\,,\\
\Gamma_\bullet\eta &\equiv \frac{1}{\sqrt{2}}(\Gamma_1-i\Gamma_3)\,\eta = \sqrt{2}\,e_1\wedge\eta\,, \qquad & \Gamma_{\bar\bullet}\eta &\equiv \frac{1}{\sqrt{2}}(\Gamma_1+i\Gamma_3)\,\eta = \sqrt{2}\,e_1\rfloor\eta\,.
\end{aligned} \tag{B.7}$$

The Clifford algebra relations in this basis are $\{\Gamma_A,\Gamma_B\} = 2\eta_{AB}$, where $A, B, \ldots = +, -, \bullet, \bar\bullet$ and the nonvanishing components of the tangent space metric read $\eta_{+-} = \eta_{-+} = \eta_{\bullet\bar\bullet} = \eta_{\bar\bullet\bullet} = 1$. The spinor 1 is a Clifford vacuum, $\Gamma_+1 = \Gamma_{\bar\bullet}1 = 0$, and the representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^+ = \Gamma_-, \Gamma^{\bar\bullet} = \Gamma_\bullet$, so that any spinor can be written as

$$\eta = \sum_{k=0}^{2} \frac{1}{k!}\phi_{\bar a_1\ldots\bar a_k}\Gamma^{\bar a_1\ldots\bar a_k}1\,, \qquad \bar a = +, \bar\bullet\,.$$

The action of the Gamma matrices and the Lorentz generators Γ_{AB} is summarized in table 2.

Note that $\Gamma_A = U_A{}^a\Gamma_a$, with

$$(U_A{}^a) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0\\ -1 & 0 & 1 & 0\\ 0 & 1 & 0 & -i\\ 0 & 1 & 0 & i \end{pmatrix} \in \mathrm{U}(4)\,,$$

so that the new tetrad is given by $E^A = (U^*)^A{}_a E^a$.

[15]It is known that on even-dimensional manifolds there are two Spin invariant Majorana inner products. The other possibility, based on $C = i\Gamma_{03}$, was used in [26].

	1	e_1	e_2	$e_1 \wedge e_2$
Γ_+	0	0	$\sqrt{2}$	$-\sqrt{2}e_1$
Γ_-	$\sqrt{2}e_2$	$-\sqrt{2}e_1 \wedge e_2$	0	0
$\Gamma_\bullet$	$\sqrt{2}e_1$	0	$\sqrt{2}e_1 \wedge e_2$	0
$\Gamma_{\bar{\bullet}}$	0	$\sqrt{2}$	0	$\sqrt{2}e_2$
Γ_{+-}	1	e_1	$-e_2$	$-e_1 \wedge e_2$
$\Gamma_{\bar{\bullet}\bullet}$	1	$-e_1$	e_2	$-e_1 \wedge e_2$
$\Gamma_{+\bullet}$	0	0	$-2e_1$	0
$\Gamma_{+\bar{\bullet}}$	0	0	0	2
$\Gamma_{-\bullet}$	$-2e_1 \wedge e_2$	0	0	0
$\Gamma_{-\bar{\bullet}}$	0	$2e_2$	0	0

Table 2: The action of the Gamma matrices and the Lorentz generators Γ_{AB} on the different basis elements.

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